2007

QCD resummation for the fully differential Drell-Yan cross section

Ricardo A. Rodriguez-Pedraza
Iowa State University

QCD resummation for the fully differential Drell-Yan cross section

by

Ricardo A. Rodriguez-Pedraza

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: High Energy Physics

Program of Study Committee:
Jianwei Qiu, Major Professor
James Cochran
John Lajoie
German Valencia
Ananda Weerasinghe

Iowa State University

Ames, Iowa

2007

DEDICATION

Ad Beatam Vírginem Mariam Mediatricem

I would also like to dedicate this work to my wife Carolina, and my daughter Camila. Without their support and loving guidance I would have not been able to complete this work.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

We study an extension of resummation to the fully differential cross section in the Drell-Yan process. This new method extends the Collins-Soper- Sterman formalism to the longitudinal W_L and double delta helicity structure function $W_{\Delta\Delta}$, recovering the next-to-leading-order predictions. The new extension also modifies the transverse structure function W_T obtained in previous extensions.

The angular coefficients, λ and ν, used for parametrization of the angular distribution, were studied with the new structure functions. No violation of the Lam-Tung relation was found. A possible solution to explain the difference between theoretical and experimental results is proposed. This solution may also explain the existence of the azimuthal asymmetry.

For completeness, leading-order and next-to-leading-order results are presented. The Collins-Soper-Sterman formalism is also reviewed.

CHAPTER 1. INTRODUCTION

The Drell-Yan (DY) process also known as lepton pair production is an inclusive large momentum transfer reaction where two hadrons collide to produce a lepton pair coming from the decay of a massive vector boson V. This reaction is sometimes schematically written as:

$$h(P_A) + h'(P_B) \rightarrow V(q) + X \rightarrow l_- + l_+ + X$$

where X includes all undetected final hadron states.

This type of process was first seen at BNL by Christenson, *et al.* [34] and [35]. They studied the collision

$$p + U \rightarrow \mu^+\mu^- + X$$

for proton energies between 22 to 29 GeV and muon pair mass of around 1.7 GeV. The spectrum of lepton pair production observed by them, and many others after, is composed by the superposition of the continuum which is explained through the DY mechanism [59], [60], [126] and some quarkonium states which allowed the discovery of the charm quark and the beauty quark in the 1970's. For example, the J/Ψ was discovered [1] by muon pair production at BNL [8] and later on in 1977 the Υ family of resonances was observed at Fermilab [79].

By 1980 DY was already providing information about the antiquark structure of the nucleon [95], and by combining data obtained for the parton distributions of the proton and antiproton the valence and sea distributions inside of both particles can be obtained [10]. It was also possible, for the first time, to find out the distributions of unstable particles like the

[1] This discovery was simultaneous with the e^+e^- experiment at SLAC [9]

pion and kaon [11], [99]. It is also worth to mention the discovery of $W^{\pm}$ and Z^0 and the role that DY played.

Since the 1990's DY has become together with deep inelastic scattering (DIS) as an important source in the global fits for the parton distributions inside a nucleon [104]. DY data has recently provided the first measurement of the x dependence of the ratio $\overline{d}/\overline{u}$ [100] and has been part of the search for exotic particles, like the Z', using forward-backward asymmetry [110] and the detection of extra dimensions [80]. In the new millennium DY remains a fertile field for theory [22], [66] and experiment [56], [122].

The angular distribution of the leptons also offers some interesting surprises. When the vector boson is a virtual photon we can write this distribution (Sec.2.3):

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega} = \left[\frac{3}{4\pi}\frac{1}{\lambda+3}\right]\left[1+\lambda\cos^2\theta+\mu\sin 2\theta\cos\phi+\left(\frac{\nu}{2}\right)\sin^2\theta\cos 2\phi\right]$$

where the coefficients λ, μ and ν may in general depend on the kinematical variables of the process and $d\Omega = d\cos\theta d\phi$ with θ and ϕ polar and azimuthal angles measured in the vector boson's rest frame (Sec. 2.1). The parton model predicts $\lambda = 1$ $\mu = \nu = 0$. Experimentally these predictions have been tested in three different ranges of energy and two different systems: NA10 collaboration used $\pi^- + W$ at 194 GeV/c [67], [77], E615 collaboration used the same system with 252 GeV/c [53] and E886 collaboration used $p + d$ at 800 GeV/c [128]. The experimental data showed ν as large as 0.3 [67] and [53], this phenomenon is known as the $\cos 2\phi$ asymmetry or azimuthal asymmetry in unpolarized Drell-Yan. The asymmetry was found to be independent of nuclear corrections, increasing with the transverse momentum of the lepton pair. The values for λ and μ coincide with the theoretical predictions except for a few cases , [67], [77]. It is important to mention here that E615 results consistently exhibit much larger values of ν [128]. Meanwhile the recent results from E886 show ν consistent with zero [128]. In Fig. 1.1 taken from [128] we can observe the experimental results of the three collaborations together with fits to the data using:

$$\nu = 16\kappa_1 \frac{p_T^2 M_C^2}{\left(p_T^2 + 4M_C^2\right)^2} \tag{1.1}$$

where M_C is a constant with value of about 2.4 GeV/c^2 and $\kappa_1 = 0.47 \pm 0.14$ for NA10, $\kappa_1 = 0.93 \pm 0.10$ for E615 and $\kappa_1 = 0.11 \pm 0.04$ for E866 [128].

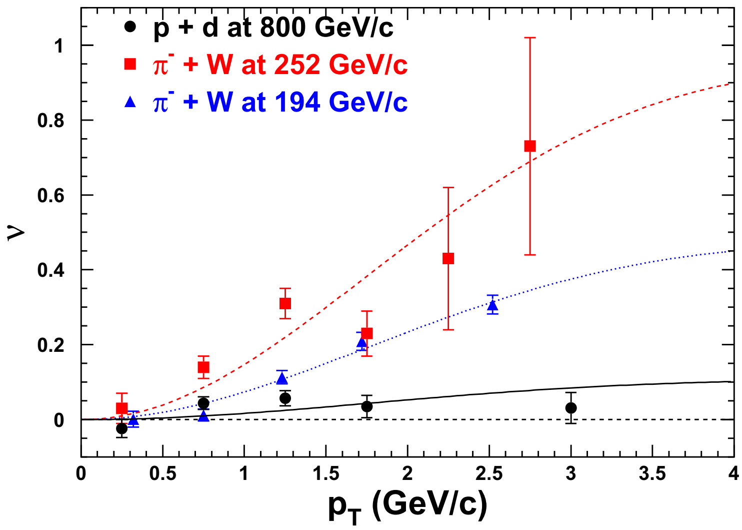


Figure 1.1 ν vs p_T.

Continuing with the surprises, Lam and Tung [88] deduced for the DY process an analogue of the Callan-Gross relation of DIS which expressed in terms of λ, μ, ν reads:

$$2\nu - (1 - \lambda) = 0$$

This relation assumes massless and unpolarized spin 1/2 partons and neglects their intrinsic transverse momentum. The Lam-Tung formula just states that at high energies the dominant cross section is for the production of a virtual photon with transverse polarization and it is valid in any frame where the lepton pair is at rest [38], [88].

Experiments show two types of results. The NA10 and E866 are largely consistent with the Lam-Tung relation, while the E615 clearly establishes the violation of this relation [128].

(See Fig. 1.2 taken from [128].)

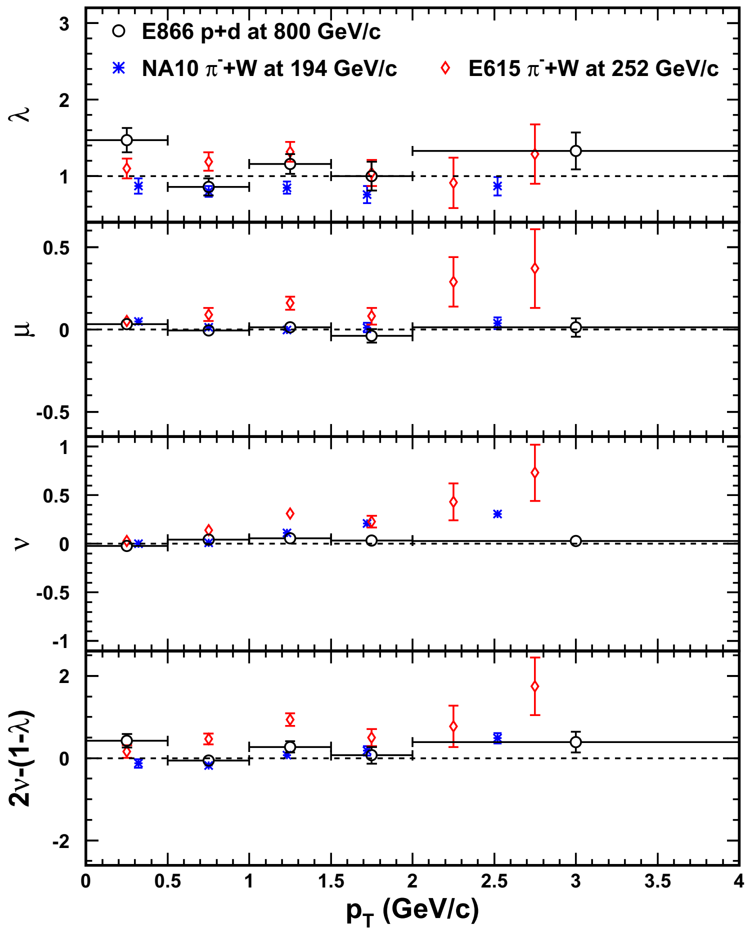


Figure 1.2 Parameters λ, μ, ν and $2\nu - (1 - \lambda)$ vs p_T.

Several types of corrections to the naive picture of the QCD modified parton model have been put forward in order to explain the violation of the Lam-Tum sum rule and the $\cos 2\phi$ asymmetry. We can generically classify these corrections into two types: perturbative and higher twist.

Higher order corrections in α_S change the predicted values for λ, μ and ν but the corrections at next to leading order do not alter the Lam-Tung relation [88] and it is almost unchanged by next to next to leading order. It was also found that the slight violation predicted has the wrong sign compared with experiment [96].

Cleymans and Kuroda [37], [38] showed that ν and the Lam-Tung relation are modified by the presence of intrinsic transverse momentum. Unfortunately the value of such modification for ν is no bigger than 0.005 when $\nu \approx 0.290$ [67] and 0.05 for the Lam-Tung rule [53].

One particular model of higher twist corrections proposed very early [15],[16] and developed further in [28], [65] considers non-scaling, non-factoring $1/Q^2$ contributions that assume that the number of partons participating in the initial state of DY are more than the minimum necessary. One of the quarks is assumed far off-shell and therefore needs to be regarded as bound. The bound state is characterized by a gluon exchange with the quark that does not participate in the hard scattering. Results from [28] allow to conclude that the violation of the Lam-Tum sum rule and the azimuthal asymmetry may not be fully explained by higher twist effects of the type just described. It is worth to notice that this model has not been completely ruled out.

In 1993, Brandenburg *et al.* [27], proposed that a nonperturbative gluonic background could produce factorization breaking spin correlations in the initial state of the partons. Based in this idea they proposed $1 - \lambda - 2\nu \approx -4\kappa$ where κ is a measure of the correlation between the transverse spins of the incoming quarks with

$$\kappa = \kappa_0 \frac{Q_T^4}{Q_T^4 + m_T^4} \tag{1.2}$$

this simple ansatz fits the 194 GeV/c data of NA10.

Boer and Mulders [19] advanced a mechanism inside the factorization frame in order to elucidate single spin asymmetries in the DY process and in $pp^{\uparrow} \to \pi X$. Their idea is the existence of a transversity distribution function $h_1^{\perp}$ that is chiral-odd T-odd with intrinsic transverse momentum dependence. This function can be interpreted as the distribution of a transversely polarized quark with nonzero transverse momentum inside an unpolarized hadron. Boer later used the same function to explain a nonzero κ [20]. Since then, some models have been suggested to explain the possible origin of $h_1^\perp$. For example in [22], $h_1^\perp$ of the proton and the resulting $\cos 2\phi$ asymmetry were found in a quark-scalar diquark model with initial state gluon interaction. Lu and Ma [94] did similar calculations but this time $h_1^\perp$ of the pion was determined in a quark-spectator-antiquark model with final state interaction. They were able to fit the data in a reasonable way.

At this moment it is important to mention the compatibility between the ansatz given by Brandenburg *et al.* [27] and the ideas advanced by Boer [20]. Of course some restrictions are necessary in the general approach of [27], factorization is the most significant among others. The authors of both papers also have suggested one more possibility as source of spin correlations: instantons [23].

Since the large ν values observed by NA10 and E615 are absent in $p+d$ and the Lam-Tung relation remains valid also in this system there are constraints on theoretical models that predict a large azimuthal asymmetry originating from QCD vacuum effects. The experimental results also suggest that the Boer-Mulders function for sea quarks is much more smaller than that for valence quarks [101].

So far we have left out the effects of soft gluon emission in the violation of the Lam-Tum sum rule and the $\cos 2\phi$ asymmetry. Chiappetta and Le Bellac [33] considered soft-gluon resummation at low Q_T in impact parameter space in the Collins Soper (CS) frame [39] following the formalism of Altarelli *et al.* [2]. They were unable to reproduce the experimental behavior of ν and found that the deviation of the angular distributions from the $1 + cos^2\theta$ naive behavior is not greater that 5%. Balázs *et al.* [12] and Ellis *et al.* [63] applied the Collins-Soper-Sterman resummation method (CSS) [44], to the analysis of the decay of angular distributions for electroweak vector bosons in the CS frame. All the above authors have only resummed the dominant terms of the form $\alpha_S^k \ln\left(Q^2/Q_T^2\right)/Q_T^2$ which are only present in the transverse component of the angular distribution and kept the leading order expressions for the other coefficients. This approach ignores the existance, in some of the angular factors, of the $1/Q_T$ divergence and large logarithmic corrections as $Q_T \rightarrow 0$.

Boer ([21] and [24]) using the CSS resummation formalism and the transversity distribution function $h_1^{\perp}$ has shown the importance of the nonperturbative Sudakov factor to explain the Q^2 behavior of the $\cos 2\phi$ asymmetry. Gamberg and Goldstein [70] performed a similar analysis using factorization and the transversity distribution predicting the dependence of ν as a function of the transverse momentum and the invariant mass of the lepton pair. Quite recently Boer and Vogelsang [25] have revisited the role of resummation in DY at small transverse momentum; they made clear that the angular coefficients are frame dependent and that when there is a change of frame the logarithmic terms get reshuffled among them.

Here, we will calculate the fully differential DY cross section resumming the scalar functions of the hadron tensor in a frame independent way. This resummation is inspired by the CSS resummation formalism. We will discuss also how this extension may be a possible explanation for the violation of the Lam-Tum sum rule and the azimuthal asymmetry. This study is also relevant for processes like semi-inclusive deep inelastic scattering and back-to-back hadron production in two-jet events in electron-positron annihilation, once the definitions of frames and coordinate axes have been done and where no angular distributions exist.

This dissertation is organized as follows: Chapter 2 contains the basic definitions of the kinematic variables involved in the process, together with a general analysis of the cross section assuming only the decay of a heavy photon together with the introduction of the structure functions; quantities that are measured in experimental setups. Chapter 3 describes the picture of the DY process in the parton model and the QCD "corrections". In Chapter 4 the low Q_T limit is taken so resummation appears as consequence of the factorization in this region of the phase space. In Chapter 5 the possible extensions of resummation to the fully differential cross section appear and numerical results and conclusions are also shown.Three appendices complete the thesis. Appendix A contains some extra reference frames used in the literature, Appendix B shows the QCD corrections to the Drell-Yan picture and Appendix C has several mathematical results included in order to have a self-contained explanation.

CHAPTER 2. MODEL INDEPENDENT CONSIDERATIONS

A general definition of the Drell-Yan cross section is presented here. In order to describe this cross section it is necessary to introduce two different reference frames: the hadron center-of-mass system and the dilepton center-of-mass system. The corresponding kinematic variables are also defined. Several types of structure functions are presented in order to exhibit alternative ways to describe the different components of the cross section.

2.1 Kinematics

In order to describe the Drell-Yan process we will use two coordinate frames: center-of-mass system of the incident hadrons, or just hadron c.m.s and the center-of-mass system of the two leptons also known as the dilepton c.m.s. In the hadron c.m.s the hadrons are collinear and the Z-axis is chosen along the beam direction; the X-axis is chosen to be in the direction of the transverse momentum of the massive boson[1] and the Y-axis just follows from the right hand rule. We will denote the components in the hadron c.m.s as:

- P_A^μ beam momentum
- P_B^μ target momentum
- l_-^μ negative lepton momentum
- l_+^μ positive lepton momentum
- $q^\mu \equiv l_+^\mu + l_-^\mu$ for the momentum of massive boson

[1]Note that this implies that in absence of Q_T the X-axis and Y-axis are undefined

We will work with massless particles $P_A^2 = P_B^2 = l_\pm^2 = 0$. This means that we are assuming two conditions: the square of the hadronic center-of-mass energy is much bigger that the hadron masses $S = (P_A + P_B)^2 \gg P_A^2, P_B^2$ and the invariant mass of the dilepton is much bigger that the lepton masses $q^2 = Q^2 \gg l_\pm^2$. Experimentally it is possible to measure the two momenta of the produced leptons except in the case of the production of $W^\pm$ boson. With this information five Lorentz-invariant quantities[2] can be found:

- Q^2 the invariant mass of the vector boson V
- y the rapidity of V
- Q_T^2 transverse momentum square of V
- θ, ϕ polar and azimuthal angles of the positive lepton defined in the dilepton c.m.s.

Now, we need to specify the axes in the dilepton c.m.s. The only condition that we have so far is that the boson should be at rest in this frame and since $Q^2 > 0$ this is always possible. In general if $Q_T \neq 0$ the beam momentum $\vec{P}_A$ and target momentum $\vec{P}_B$ are not collinear in the dilepton c.m.s and therefore they define a plane, the $(\vec{P}_A, \vec{P}_B)$ plane. We will demand that the y-axis be perpendicular to this plane[3] and parallel to the Y-axis of the hadron c.m.s [4]. We will require also that when $Q_T = 0$ the z-axis should be the same for both frames [5]. Note that the direction of the x-axis is given by the right hand rule after we have chosen the z-axis. As soon as we have selected our particular frame we can transform to any similar set of Cartesian coordinates through a rotation around the y-axis [6]. Some of the popular choices for z-axis found in the literature are [53], [67], [86] :

- $\hat{z}$ parallel to the bisector of $\vec{P}_A$ and the negative of the target momentum $-\vec{P}_B$, this is the *Collins-Soper* (CS) frame [39],
- $\hat{z}$ parallel to the beam momentum $\vec{P}_A$, this is the *t-channel helicity* or *Gottfried-Jackson* (GJ) frame [71],

[2] The Lorentz invariance is with respect to boosts along the z-axis
[3] Therefore y-axis is now normal to the reaction plane
[4] This is the convention of [39] but antiparallel to [53],[67].
[5] Thus when $Q_T = 0$ the two systems will differ only by a boost along the common z-axis

- $\hat{z}$ antiparallel to the target momentum $\vec{P}_B$, this is the *u-channel* (UC) frame [67],

- $\hat{z}$ antiparallel to the sum of beam momentum and target momentum $\vec{P}_A + \vec{P}_B$, this is the *s-helicity* (SH) frame [58].

Thus the polar angle θ is the angle between $\hat{z}$ and $\vec{l}_+$ and the azimuthal angle ϕ is the angle between the $\left(\vec{P}_A, \vec{P}_B\right)$ plane and the $\left(\hat{z}, \vec{l}_+\right)$ plane, see for example the angles in Fig. 2.1.

Here, we are only going to describe the CS frame leaving the depiction of the other Cartesian systems to the Appendix A. The reason of this choice is based in the behavior of this frame at low Q_T. In this region the CS frame produces the simplest expressions for the helicity structure functions (see Sec.2.3) because it minimizes the effects of the internal transverse momentum of the colliding partons [86]. Another reason is the smooth transition of the kinetic variables and helicity structure functions defined in this frame in the limit $Q_T = 0$.

To reach the CS frame for the hadron c.m.s we can follow the next two steps. First boost along the $\hat{z}$-axis to an intermediate frame O^* in which $Q_z^* = 0$. Then a second boost in the $-Q_T$ direction. In this frame as in the others $\vec{Q}' = 0$ and $Q_0^{'} = Q$. Thus the matrix of transformation of coordinates from hadron c.m.s to CS is given by:

$$\Lambda_{CM \to CS} = \begin{pmatrix} \frac{Q_0}{Q} & -\frac{Q_T}{Q} & 0 & -\frac{Q_z}{Q} \\ -\frac{Q_0 Q_T}{Q\sqrt{Q^2+Q_T^2}} & \frac{\sqrt{Q^2+Q_T^2}}{Q} & 0 & \frac{Q_z Q_T}{Q\sqrt{Q^2+Q_T^2}} \\ 0 & 0 & 1 & 0 \\ -\frac{Q_z}{\sqrt{Q^2+Q_T^2}} & 0 & 0 & \frac{Q_0}{\sqrt{Q^2+Q_T^2}} \end{pmatrix} \tag{2.1}$$

where $(Q_0, Q_T, 0, Q_z)$ are the components of q^μ measured in the hadron c.m.s.
The vectors $\vec{P}_A^{'}$ and $\vec{P}_B^{'}$ now make equal angles with the $\hat{z}$-axis, $\beta = \arctan\left(Q_T/Q\right)$. In this way the definition of the transverse axes in the CS frame are determined by those of the hadron c.m.s [39], see also Fig.2.1.

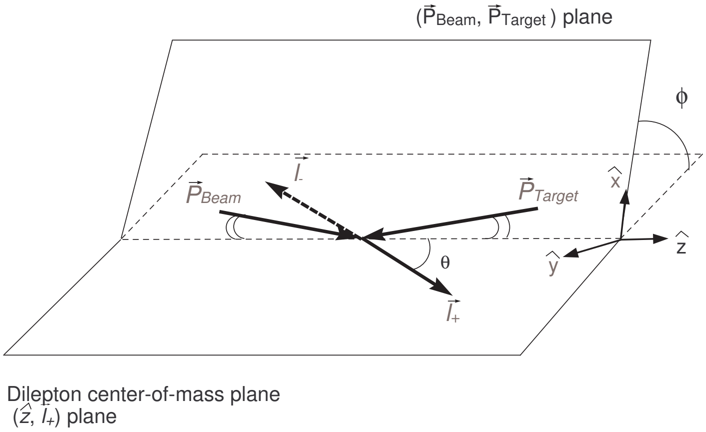


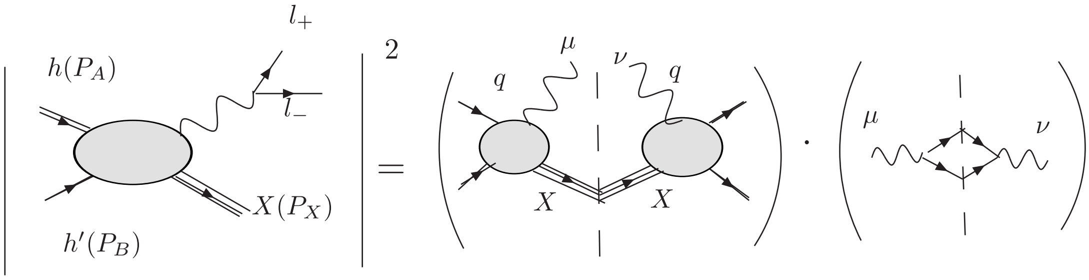
Figure 2.1 The Collins-Soper frame

2.2 The Drell-Yan cross section

The DY process is given by the elementary amplitude

$$\bar{u}(l_+)\,(ie\gamma_\mu)\,v(l_-)\frac{i\,(-g^{\mu\nu})}{(l_+ + l_-)^2}\,\langle X|\,eJ_\nu(0)\,|P_A, S_A; P_B, S_B\rangle \tag{2.2}$$

were we are assuming that the interaction is electromagnetic, so the massive vector boson is a virtual photon γ^* with momentum q. The square of the amplitude can be represented by the following diagrams



These diagrams already make explicit that we can separate the leptonic and hadronic degrees of freedom using two independent tensors to write the square of the amplitude.

With this information in mind we can write the DY cross section

$$d\sigma = \frac{e^4}{2S^2}\frac{d^3l_+}{(2\pi)^3 2E_+}\frac{d^3l_-}{(2\pi)^3 2E_-}\frac{1}{(l_+ + l_-)^4}L^{\mu\nu}W_{\mu\nu} \tag{2.3}$$

the leptonic tensor $L^{\mu\nu}$ is given by (neglecting the masses of the leptons),

$$\begin{aligned} L^{\mu\nu} &\equiv \frac{1}{2} Tr\left[\gamma^{\mu} \not{l}_{+} \gamma^{\nu} \not{l}_{-}\right] \\ L^{\mu\nu} &= 2 l_{-}^{\mu} l_{+}^{\nu} - Q^2 g^{\mu\nu} + 2 l_{+}^{\mu} l_{-}^{\nu} \end{aligned} \tag{2.4}$$

and the hadronic tensor is the square of the hadron matrix element shown in Eq.(2.2)

$$\begin{aligned} W_{\mu\nu} \equiv \quad & S \sum_{X, P_X} (2\pi)^4 \delta^4(P_A + P_B - l_+ - l_- - P_X) \\ & \times \quad \langle X(P_X)| J_\mu(0) |P_A P_B\rangle^* \langle X(P_X)| J_\nu(0) |P_A P_B\rangle \end{aligned} \tag{2.5}$$

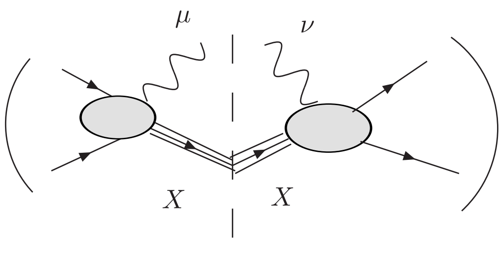
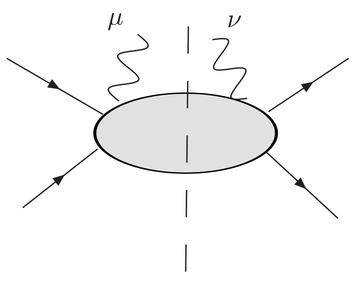
Note that an average over the spins of the initial hadron states is understood. Using the completeness relation $\sum_{X,P_X} |X(P_X)\rangle \langle X(P_X)| = 1$ and the translation invariance,

$$J^\dagger_\mu(y) = e^{i(\hat{P}_A + \hat{P}_B)\cdot y} J^\dagger_\mu(0) e^{-i\hat{P}_X \cdot y}$$

it is possible to rewrite $W_{\mu\nu}$ as an expectation value of a bilocal operator[6] [86]:

$$W_{\mu\nu} = S \int d^4y \; e^{i(l_+ + l_-)\cdot y} \langle P_A P_B| J^\dagger_\mu(y) J_\nu(0) |P_A P_B\rangle \tag{2.6}$$

It is easy to see from this expression that the hadronic tensor contains the dynamical information of the hadron state as probed by the virtual photon. Graphically:

$\sum_{X,P_X}$  $=$ 

In order to completely separate the lepton from the hadron degrees of freedom we can introduce $1 = \int d^4q \delta^4(l_+ + l_- - q)$ in the phase space of the DY cross section

$$\begin{aligned} \frac{d^3 l_+}{(2\pi)^3 2E_+} \frac{d^3 l_-}{(2\pi)^3 2E_-} &= d^4q \frac{d^3 l}{(2\pi)^6 4E^2} \delta(Q - 2E) \\ &= \frac{d^4q d\Omega dE}{8(2\pi)^6} \delta(\frac{Q}{2} - E) \\ &= \frac{d^4q d\Omega}{8(2\pi)^6} \end{aligned} \tag{2.7}$$

[6]There are several other normalization conventions for the hadronic tensor, see for example [14], [98], [121]. The advantage here is a dimensionless tensor.

where $d\Omega \equiv d\cos\theta d\phi$. Equation (2.7) allows us to rewrite the DY cross section (2.3) as:

$$\frac{d\sigma}{d^4q d\Omega} = \frac{\alpha^2}{2S^2Q^4\,(2\pi)^4} L^{\mu\nu} W_{\mu\nu} \tag{2.8}$$

here the fine structure constant is given in natural units $\alpha = \frac{e^2}{4\pi}$. Now the lepton tensor is only function of q and l: $L^{\mu\nu}(l_+, l_-) \rightarrow L^{\mu\nu}(l, q)$ and similarly for the hadron part $W_{\mu\nu}(P_A, P_B, l_+, l_-) \rightarrow W_{\mu\nu}(P_A, P_B, q)$, thus the separation is complete and manifest[7].

2.3 Structure functions

The Lorentz tensor $W_{\mu\nu}(P_A, P_B, q)$ can be written as a sum of products of tensors and scalar functions called *structure functions*. In principle we have at our disposition several possibilities that may include combinations of the following terms $g^{\mu\nu}$, $P_A^\mu P_A^\nu$, $P_B^\mu P_B^\nu$, $P_A^\mu P_B^\nu$, $P_A^\mu q^\nu$, $P_B^\mu q^\nu$ and $q^\mu q^\nu$ but symmetry considerations enter into play [8]. From the properties of the electromagnetic current and strong interactions we have some requirements:

$$q_\mu W^{\mu\nu} = 0 \qquad \text{Current conservation, gauge invariance} \tag{2.9}$$

$$(W^{\nu\mu})^* = W^{\mu\nu} \qquad \text{Hermiticity} \tag{2.10}$$

$$W_{\mu\nu}(\bar{P}_A, \bar{P}_B, \bar{q}) = W^{\mu\nu}(P_A, P_B, q) \qquad \text{Parity} \tag{2.11}$$

$$W_{\mu\nu}(\bar{P}_A, \bar{P}_B, \bar{q}) = [W^{\mu\nu}(P_A, P_B, q)]^* \qquad \text{Time reversal} \tag{2.12}$$

Then, for example, hermiticity requires the symmetric part of the hadronic tensor to be real. Since $L_{\mu\nu}$ is symmetric we can safely assume that $W^{\mu\nu}$ is symmetric and real. By current conservation $W^{\mu\nu}$ also is "perpendicular" to the vector q, this reduces the number of independent

[7] Note that this separation is only possible because we have used the dilepton rest frame

[8] Factors including γ^μ are missing since we are considering a tensor where configurations over spins have already been summed and averaged.

structure functions to four[9]. Thus, we can write in the most general way,

$$\begin{aligned}
W^{\mu\nu}\left(P_A, P_B, q\right) = &-\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{Q^2}\right) W_1 + \left(P_A^\mu - \frac{q\cdot P_A}{Q^2} q^\mu\right)\left(P_A^\nu - \frac{q\cdot P_A}{Q^2} q^\nu\right)\frac{W_2}{S} \\
&+ \left(P_B^\mu - \frac{q\cdot P_B}{Q^2} q^\mu\right)\left(P_B^\nu - \frac{q\cdot P_B}{Q^2} q^\nu\right)\frac{W_4}{S} \\
&- \left[\left(P_A^\mu - \frac{q\cdot P_A}{Q^2} q^\mu\right)\left(P_B^\nu - \frac{q\cdot P_B}{Q^2} q^\nu\right) + \left(P_B^\mu - \frac{q\cdot P_B}{Q^2} q^\mu\right)\left(P_A^\nu - \frac{q\cdot P_A}{Q^2} q^\nu\right)\right]\frac{W_3}{S}
\end{aligned} \tag{2.13}$$

where the structure functions are now functions of the four independent scalar invariants $W_i = W_i\left(q^2, S, q\cdot P_A, q\cdot P_B\right)$. Note, that the gauge invariance and the symmetry of $W^{\mu\nu}$ are explicit. The structure functions shown here are known as *invariant structure functions* [10].

For reasons of theoretical convenience [86], it is useful to introduce *helicity structure functions.* These functions are defined as the contraction of the hadronic tensor with a set of polarization vectors defined with respect to one of the dilepton rest frames:

$$W_{\lambda,\lambda'} \equiv \epsilon_\lambda^\mu W_{\mu\nu} \epsilon_{\lambda'}^{*\nu} \tag{2.14}$$

here the ϵ_λ^μ are the polarization vectors of the virtual photon with:

$$\begin{aligned}
\epsilon_0^\mu &\equiv \hat{z}^\mu \\
\epsilon_{\pm 1}^\mu &\equiv \left(\mp\hat{x} - i\hat{y}\right)^\mu / \sqrt{2}
\end{aligned}$$

and they also satisfy:

$$\begin{aligned}
q_\mu \epsilon_\lambda^\mu &= 0 \\
\epsilon_\lambda^{*\mu} \epsilon_\lambda^\nu g_{\mu\nu} &= -1
\end{aligned} \tag{2.15}$$

where the vectors $(\hat{x}, \hat{y}, \hat{z})$ are the unit vectors of the respective Cartesian set in the chosen rest frame. These vectors can be expressed in terms of the components of q^μ using the appropriate inverse matrix of any of the following transformations: (2.1), (A.1), (A.2), (A.3).

[9] When J^μ does not respect parity the number of independent functions rises to 9

[10] This definition of the invariant structure functions is by no means unique. See for example [86] and [121]

Let us denote the helicity structure functions following the next set of conventions [86]:

$$
\begin{aligned}
W_L &\equiv W_{0,0} \\
W_T &\equiv W_{1,1} \\
W_{\Delta\Delta} &\equiv \frac{W_{1,-1} + W_{-1,1}}{2} \\
W_\Delta &\equiv \frac{W_{1,0} + W_{0,1}}{\sqrt{2}}
\end{aligned}
\tag{2.16}
$$

Thus, as their respective name indicates, W_T is the structure function for a virtual photon with transverse polarization, W_L is for longitudinal polarization, W_Δ is for a single-spin flip and $W_{\Delta\Delta}$ is for a double-spin flip.

We can rewrite the hadronic tensor $W^{\mu\nu}$ in terms of the vectors $(\hat{x}, \hat{y}, \hat{z})$ and the helicity structure functions:

$$
\begin{aligned}
W^{\mu\nu} = &- \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{Q^2}\right)(W_T + W_{\Delta\Delta}) - 2\hat{x}^\mu \hat{x}^\nu W_{\Delta\Delta} + \hat{z}^\mu \hat{z}^\nu (W_L - W_T - W_{\Delta\Delta}) \\
&- (\hat{x}^\mu \hat{z}^\nu + \hat{x}^\nu \hat{z}^\mu) W_\Delta
\end{aligned}
\tag{2.17}
$$

From (2.17) is easy to deduce

$$
W^{\mu\nu}(-g_{\mu\nu}) = 2W_T + W_L \tag{2.18}
$$

which reflects the two transverse polarizations of the virtual photon.

Contracting (2.17) with the lepton tensor $L_{\mu\nu}(l_+, l_-)$ in the dilepton c.m.s. where

$$
\begin{aligned}
l_+^\mu &= \frac{Q}{2}(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\
l_-^\mu &= \frac{Q}{2}(1, -\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta)
\end{aligned}
\tag{2.19}
$$

we can also express the DY cross section (2.8) in terms of the helicity structure functions [86]:

$$
\frac{d\sigma}{d^4q d\Omega} = \frac{\alpha^2}{2S^2Q^2(2\pi)^4}\left[W_T\left(1+\cos^2\theta\right) + W_L\left(1-\cos^2\theta\right) + W_{\Delta\Delta}\cos 2\phi \sin^2\theta + W_\Delta \sin 2\theta \cos\phi\right] \tag{2.20}
$$

There are several ways to extract the values of the helicity structure functions: directly from the hadron tensor using the definitions (2.16), using the following projection operators:

$$
\begin{aligned}
W_L &= W^{\mu\nu}\hat{z}_\mu\hat{z}_\nu \\
W_T &= \frac{W^{\mu\nu}\left(-g_{\mu\nu}\right)-W_L}{2} \\
W_{\Delta\Delta} &= W_T - W^{\mu\nu}\hat{x}_\mu\hat{x}_\nu \\
W_\Delta &= -W^{\mu\nu}\hat{z}_\mu\hat{x}_\nu
\end{aligned}
\tag{2.21}
$$

or they also can be extracted from the DY cross section.

In order to compare with experimental results we will introduce the *angular differential cross section* which is defined as the ratio of differential cross sections [39]:

$$\frac{dN}{d\Omega} \equiv \frac{d\sigma}{d^4q\,d\Omega}\left(\frac{d\sigma}{d^4q}\right)^{-1} \tag{2.22}$$

which is equal to:

$$\frac{dN}{d\Omega} = \frac{3}{8\pi}\frac{W_T\left(1+\cos^2\theta\right)+W_L\left(1-\cos^2\theta\right)+W_{\Delta\Delta}\cos 2\phi\sin^2\theta+W_\Delta\sin 2\theta\cos\phi}{2W_T+W_L} \tag{2.23}$$

where we have used,

$$\frac{d\sigma}{d^4q} = \frac{\alpha^2}{12S^2Q^2\pi^3}\left(2W_T+W_L\right) \tag{2.24}$$

We can also rewrite the angular differential cross section as [53], [67]:

$$\frac{dN}{d\Omega} = \frac{3}{4\pi}\frac{1}{\lambda+3}\left(1+\lambda\cos^2\theta+\mu\sin 2\theta\cos\phi+\frac{\nu}{2}\cos 2\phi\sin^2\theta\right) \tag{2.25}$$

the relation between $W_T, W_L, W_\Delta, W_{\Delta,\Delta}$ and λ, μ, ν can be easily obtained:

$$\lambda = \frac{W_T-W_L}{W_T+W_L} \qquad \mu = \frac{W_\Delta}{W_T+W_L} \qquad \nu = \frac{2W_{\Delta\Delta}}{W_T+W_L} \tag{2.26}$$

Equivalently, we can use a different parametrization [39],

$$\frac{dN}{d\Omega} = \frac{3}{16\pi}\left[1+\cos^2\theta+\left(\frac{1}{2}-\frac{3}{2}\cos^2\theta\right)A_0+2\sin\theta\cos\theta\cos\phi A_1+\frac{1}{2}\cos 2\phi\sin^2\theta A_2\right] \tag{2.27}$$

and the relations between $W_T, W_L, W_\Delta, W_{\Delta\Delta}$ and A_0, A_1, A_2 are:

$$A_0 = \frac{2W_L}{2W_T+W_L} \qquad A_1 = \frac{2W_\Delta}{2W_T+W_L} \qquad A_2 = \frac{4W_{\Delta\Delta}}{2W_T+W_L} \tag{2.28}$$

Now the labor for the theory is to calculate $W^{\mu\nu}$ to obtain the helicity structure functions. We will see in the next chapter how this is done in the parton model and how QCD corrects this naive picture.

CHAPTER 3. THE PARTON MODEL AND QCD CORRECTIONS

The Drell-Yan quark-antiquark annihilation picture for dilepton production rests on three basic assumptions: On-shell massless partons, spin 1/2 partons with no polarization if the parent hadron is unpolarized, and the coupling to the virtual photon is given by QED. We will see in this chapter how QCD generalizes the parton model. QCD predictions for the helicity functions to next-to-leading-order are also presented.

3.1 The parton model

3.1.1 The Drell-Yan picture

A light hadron is a bound state of several components where the ratio of the binding energy to the mass of the constituents is about unity [90]. Because this ratio is so high, compared with systems like the atom or the nucleus, it is not sound to suppose that the constituents inside are quasi-free and also it is not sensible to assume a fixed number of these constituents. As we shall see both affirmations are frame dependent. But first, let us called the constituent particles "partons" which we will later identify as the quarks and gluons of QCD.

Following Feynman [68], [69], we will study the collision of two hadrons in the center-of-mass frame of the colliding partons [1] where both particles are moving very fast head on. But how fast? we will assume that $E_H \gg m_H, m_p$ with E_H the energy of any hadron so we can safely neglect all the masses involved. The partons inside each hadron interact with each other and exist only in virtual states [30], [115]. Let us suppose that these virtual states have a lifetime in the rest-frame of the hadron equal to τ which has an effective lower bound $\tau_0 > 0$,

[1] The following analysis will work in any generic *infinite momentum frame*

so each hadron is made up of virtual states of non-zero lifetime.

Now, let us see the collision from the point of view of one of the participating partons. This parton will see the approaching hadron experiencing Lorentz contraction and time dilation. For instance, τ is dilated to $\tau \frac{E_H}{m_H}$ whereas the radius r_H is Lorentz contracted to $r_H \frac{m_H}{E_H}$ along the direction of motion. Therefore the colliding hadron will appear as a "pancake" with a contraction along the direction of motion while the perpendicular direction is not affected, see Fig.(3.1.a). At the moment of the collision, Fig.(3.1.b) the partons in this hadron appear "frozen" because their self-interactions act at dilated time scales that are much longer than the collision time.

Since the partons do not interact inside this hadron, there will be a single virtual state with a well defined number of constituents when the collision takes place. Each parton will carry a definite fraction ξ_i of the hadron's momentum in the center of mass-frame of the parton-parton collision and each ξ_i will satisfy $0 \leq \xi_i \leq 1$ since it is unlikely that there can exist a parton moving in opposite direction. Hence, we are assuming that the partons participating in the hard scattering come with momentum $\xi_i P^\mu$ where P^μ is the momentum of the parent hadron and $P^2 = 0$. If we also require that the parton density is not too high, the collision will essentially involve only two partons. Then it makes sense to talk of the interaction of two partons with defined momentum instead of the collision of two hadrons. After the collision Fig.(3.1.c) anything can happen but the "final-state interactions" will not interfere with the hard collision.

Thus, it is not surprising that the DY cross section in the parton model is essentially classical. This means that it is computed combining probabilities instead of amplitudes. We will introduce the parton distribution function $f_{j/H}(\xi)$ as the probability to encounter a frozen noninteracting parton of species j with momentum fraction ξ inside a hadron H. In the parton model the Drell-Yan process involves specifically the annhilation of a parton and anti-parton pair one for each hadron.

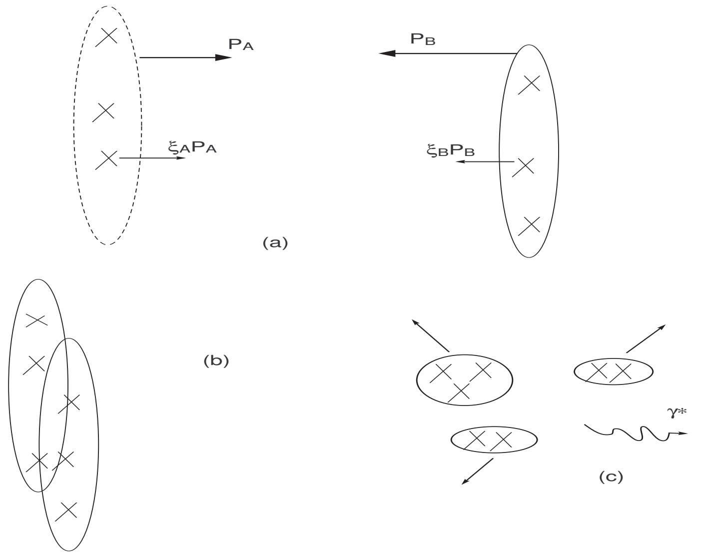
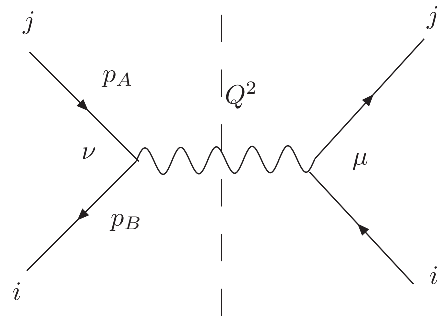


Figure 3.1 Schematic parton-model picture for the Drell-Yan process

We can relate the participating partons as a pair quark, antiquark that annihilate each other. Since our job is to calculate the hadronic tensor we need only to consider the following QED cut diagram:



which will give us the "partonic tensor" :

$$
\begin{aligned}
w^{\mu\nu}_{j\bar{j}\to\gamma^*} &= \frac{1}{3}\frac{1}{4} Tr\left[\not{p}_A \gamma^\mu \not{p}_B \gamma^\nu\right] \\
&= \frac{1}{3}\left(p_A^\mu p_B^\nu + p_A^\nu p_B^\mu - p_A \cdot p_B \, g^{\mu\nu}\right) \qquad (3.1)
\end{aligned}
$$

where lower case letters will be used to denote parton variables with

$$\begin{aligned} p_A^\mu &= \xi_A P_A^\mu \\ p_B^\mu &= \xi_B P_B^\mu \end{aligned} \tag{3.2}$$

The 1/3 factor comes from the average over initial state colors

$$\frac{1}{3} = \left(\frac{1}{3}\right)^2 \sum_{i,j=1}^{3} (\delta_{ij})^2$$

and the 1/4 comes for the average over initial spins. Therefore the hadronic tensor for the DY process is equal to:

$$W^{\mu\nu}_{h_A+h_B\to\gamma^*} = S\sum_j e_j^2 \int_0^1 \frac{d\xi_A}{\xi_A}\int_0^1 \frac{d\xi_B}{\xi_B} f_{j/A}(\xi_A)\, f_{\bar{j}/B}(\xi_B)\, w^{\mu\nu}_{j\bar{j}\to\gamma^*}\,(2\pi)^4\,\delta^4(p_A+p_B-q) \tag{3.3}$$

where e_j is the electric charge of one of the interacting quarks in units of e and the sum is over all quark and antiquark flavors. Using (2.8) we can find the corresponding cross section:

$$\frac{d\sigma_{h_A+h_B\to l^+l^-}}{d^4q d\Omega} = \sum_j \int_0^1 d\xi_A \int_0^1 d\xi_B f_{j/A}(\xi_A)\, f_{\bar{j}/B}(\xi_B)\, \frac{d\sigma_{j+\bar{j}\to l^+l^-}}{d^4q d\Omega} \tag{3.4}$$

with

$$\frac{d\sigma_{j+\bar{j}\to l^+l^-}}{d^4q d\Omega} = e_j^2 \frac{\alpha^2}{2sQ^4} w^{\mu\nu}_{j\bar{j}\to\gamma^*} L_{\mu\nu}\,\delta^4(p_A+p_B-q) \tag{3.5}$$

where $L_{\mu\nu}$ is the lepton tensor given in Eq.(2.4) and $s=(p_A+p_B)^2=\xi_A\xi_B S$.

In summary, we can consider the parton model as a generalization of the impulse approximation [30], [90]. This approximation rests upon two physical assumptions: Lorenz contraction and time dilation of internal states. The time dilation is responsible for the incoherence in the cross section, since the initial-state interactions between partons happen too early to interfere with the hard collision and final-state interactions between the fragments occur too late. An important consequence of incoherence is the universality of parton distributions, since they describe processes that depend on the hadron and are independent of the hard scattering so they are the same for all inclusive hard processes [116]. The Lorentz contraction is fundamental for the universality of the parton distribution functions, since otherwise partons from different hadrons would overlap finite times before the hard process, [2] altering the distributions [46]. Notice also that there is no interference between different flavors or different fractions ξ of the momentum.

3.1.2 Drell-Yan predictions

Our work now is to produce expressions for the helicity functions defined in Eq.(2.16). The easiest way is to find the DY cross section in the parton model. This is done contracting $w^{\mu\nu}_{j\bar{j}\to\gamma^*}L_{\mu\nu}$ in the dilepton c.m.s. The corresponding components of the lepton tensor have already been defined in the last chapter. We need then, to provide the components of $w^{\mu\nu}$. The components of the beam and target momentum are:

$$\begin{aligned} P_A &= \left(\frac{\sqrt{S}}{2}, 0, 0, \frac{\sqrt{S}}{2}\right) \\ P_B &= \left(\frac{\sqrt{S}}{2}, 0, 0, -\frac{\sqrt{S}}{2}\right) \end{aligned} \tag{3.6}$$

in the hadron c.m.s, this immediately forces the delta function in Eq.(3.5) to become $\delta^4(p_A + p_B - q) = \delta(p_A^0 + p_B^0 - Q_0)\delta(p_A^z + p_B^z - Q_z)\delta^2(\vec{Q}_T)$, so we will assume in this section that the transverse momentum of the emitted photon is zero. Consequently, using the transformation (2.1), when $Q_T = 0$, we can find the components of these two vectors in the dilepton c.m.s. :

$$\begin{aligned} P_A' &= \frac{\sqrt{S}}{2}\frac{Q_0 - Q_z}{Q}(1, 0, 0, 1) \\ P_B' &= \frac{\sqrt{S}}{2}\frac{Q_0 + Q_z}{Q}(1, 0, 0, -1) \end{aligned} \tag{3.7}$$

After some algebra the result is:

$$\frac{d\sigma_{j+\bar{j}\to l^+l^-}}{d^4q d\Omega} = e_j^2 \frac{\alpha^2}{12Q^2}\left(1 + \cos^2\theta\right)\delta(p_A^0 + p_B^0 - Q_0)\delta(p_A^z + p_B^z - Q_z)\delta^2(\vec{Q}_T) \tag{3.8}$$

where we have used the definition of the invariant mass of the photon:

$$Q^2 = Q_0^2 - Q_T^2 - Q_z^2 \tag{3.9}$$

[2] This is a type of initial-state interaction.

For reasons that will become apparent soon, we need to introduce the rapidity y of the virtual photon:

$$y = \frac{1}{2}\ln\left(\frac{Q_0 + Q_z}{Q_0 - Q_z}\right) \tag{3.10}$$

Thus the delta function and the phase space volume transform:

$$\begin{aligned} d^4q &= \frac{1}{2}dQ^2 dy d^2\vec{Q}_T && (3.11)\\ \delta(p_A^0 + p_B^0 - Q_0)\delta(p_A^z + p_B^z - Q_z) &= \frac{2}{S}\delta(\xi_A - x_A)\delta(\xi_B - x_B) && (3.12)\end{aligned}$$

with

$$\begin{aligned} x_A &= \frac{Q}{\sqrt{S}}e^{y}\\ x_B &= \frac{Q}{\sqrt{S}}e^{-y} \end{aligned} \tag{3.13}$$

so we obtain,

$$\frac{d\sigma_{j+\bar{j}\to l^+l^-}}{dQ^2 dy d^2\vec{Q}_T d\Omega} = e_j^2 \frac{\alpha^2}{12SQ^2}\left(1+\cos^2\theta\right)\delta(\xi_A - x_A)\delta(\xi_B - x_B)\delta^2(\vec{Q}_T) \tag{3.14}$$

and the corresponding DY cross section is equal to:

$$\frac{d\sigma_{h_A+h_B\to l^+l^-}}{dQ^2 dy d^2\vec{Q}_T d\Omega} = \frac{\alpha^2}{12SQ^2}\left(1+\cos^2\theta\right)\delta^2(\vec{Q}_T)\sum_j e_j^2 f_{j/A}(x_A)\, f_{\bar{j}/B}(x_B) \tag{3.15}$$

Let us pause for a second to analyze the result just obtained. We can see that the cross section (3.15) has a remarkable consequence. Integrating we find,

$$Q^4\frac{d\sigma_{h_A+h_B\to l^+l^-}}{dQ^2} = \frac{4\pi\alpha^2}{9}\frac{Q^2}{S}\sum_j e_j^2\int_0^1 d\xi_A\int_0^1 d\xi_B f_{j/A}(\xi_A)\, f_{\bar{j}/B}(\xi_B)\,\delta\left(\xi_A\xi_B - \frac{Q^2}{S}\right)$$

which explicitly shows that

$$Q^4\frac{d\sigma}{dQ^2} = \mathcal{F}\left(\frac{Q^2}{S}\right) \tag{3.16}$$

This phenomenon is known as *scaling*. It means that the cross section and the structure functions of Table 3.1 are independent of the momentum transfer Q to a certain extent. It is as if for the DY process the Q dependence is totally defined by the annihilation of the quark and anti-quark pair. This amazing result was one of the early successes of the DY model [60].

The cross section (3.16) also constitutes a complete prediction, including normalization, since the parton distribution functions are the same ones extracted from DIS. If the DY picture is correct, the transverse momentum of the lepton pair should be small, about 300 to 500 MeV and in the rest frame of the lepton pair the angular distribution with respect to the beam axis is $1+\cos^2\theta$.

We can extract from Eq.(3.15) the structure functions. Instead, we will use the projection operators defined in Eq.(2.21). Combining equations (3.1) and (3.3) we find that hadronic tensor is equal to

$$W^{\mu\nu}_{h_A+h_B\to\gamma^*} = \frac{2(2\pi)^4}{3}\sum_j e_j^2 f_{j/A}(x_A)\, f_{\bar{j}/B}(x_B)\left(P_A^\mu P_B^\nu + P_A^\nu P_B^\mu - \frac{S}{2}\, g^{\mu\nu}\right)\delta^2\left(\vec{Q}_T\right) \quad (3.17)$$

From this, it is easy to obtain for any lepton c.m.s [3]:

$$\begin{aligned}
W^{\mu\nu}(-g_{\mu\nu}) &= \frac{2(2\pi)^4}{3}S\sum_j e_j^2 f_{j/A}(x_A)\, f_{\bar{j}/B}(x_B)\,\delta^2\left(\vec{Q}_T\right) && (3.18)\\
W^{\mu\nu}\hat{z}_\mu\hat{z}_\nu &= 0 && (3.19)\\
W^{\mu\nu}\hat{x}_\mu\hat{x}_\nu &= \frac{(2\pi)^4}{3}S\sum_j e_j^2 f_{j/A}(x_A)\, f_{\bar{j}/B}(x_B)\,\delta^2(\vec{Q}_T) && (3.20)\\
W^{\mu\nu}\hat{z}_\mu\hat{x}_\nu &= 0 && (3.21)
\end{aligned}$$

so the values for the helicity structure functions are:

W_L	$=$	0
W_T	$=$	$\frac{(2\pi)^4}{3}S\,\sum_j e_j^2 f_{j/A}(x_A)\, f_{\bar{j}/B}(x_B)\,\delta^2(\vec{Q}_T)$
$W_{\Delta\Delta}$	$=$	0
W_Δ	$=$	0

Table 3.1 Parton model predictions for structure functions

One example of the behavior of W_T can be observed in Fig.3.2 together with the corresponding cross section for $p\bar{p}$ collision with a $\sqrt{S}=800$ GeV/c in the lab frame. See Fig.3.3.

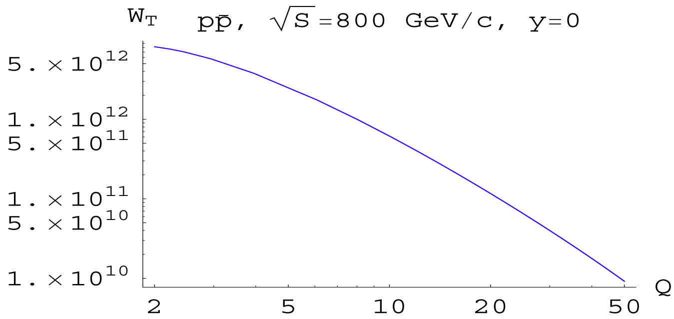


Figure 3.2 W_T vs Q, LO prediction

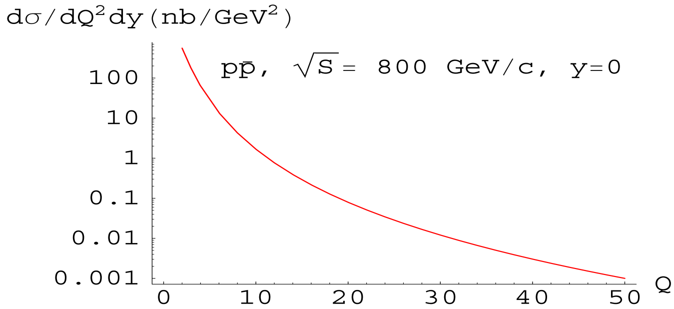


Figure 3.3 $\frac{d^2\sigma}{dQ^2dy}$ vs Q, LO prediction

From Eq.(2.26) we can express the above relations in terms of λ, μ, ν:

$$\boxed{\begin{array}{ccc} \lambda & = & 1 \\ \mu & = & 0 \\ \nu & = & 0 \end{array}}$$

Table 3.2 Parton model predictions for λ, μ and ν

We have also the interesting relations $W_L = 0$ and $W_T \neq 0$. We can understand this results as consequence of helicity conservation in QED [4]. A virtual photon with total spin 1 can only couple in a process where the quark-antiquark pair have equal helicities. Thus the photon can

[3]Remember that when $\vec{Q}_T = 0$, they are all equal.

[4]Helicity conservation is exact only for massless particles or in the high energy limit

only couple to the following two combinations[5]:

$$q_L\bar{q}_R \quad q_R\bar{q}_L$$

Let us take the second one. As it is shown in the Figure 3.4 the z-component of the spin of q_R is parallel to the direction of motion and the opposite is true for $\bar{q}_L$, the total helicity is 1. The emitted photon can only have the same helicity which means that it has circular polarization along the z'-axis or that is transversely polarized. Note that for the validity of this argument is fundamental that the quarks have spin 1/2.

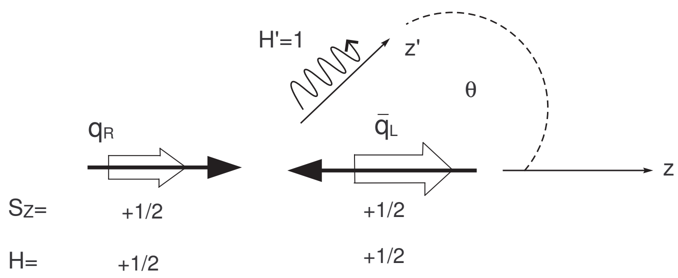


Figure 3.4 Helicity conservation in the Drell-Yan process

The parton relations of Table 3.1 imply that the helicity structure functions are not all independent as we assumed when we deduced the general structure Eq.(2.17) of the hadronic tensor. In order to find the explicit dependence let us write the most general form for the hadronic tensor in the dilepton c.m.s[6]:

$$W^{\mu\nu} = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{Q^2}\right) G_1 + \hat{x}^\mu \hat{x}^\nu G_2 + \hat{z}^\mu \hat{z}^\nu G_3 + (\hat{x}^\mu \hat{z}^\nu + \hat{x}^\nu \hat{z}^\mu) G_4 \tag{3.22}$$

and compare it with the hadronic tensor (3.3):

$$W^{\mu\nu} = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{Q^2}\right) W_T - \hat{z}^\mu \hat{z}^\nu W_T \tag{3.23}$$

[5]The value of the helicity for an antiparticle is the opposite of the corresponding value for a particle

[6]To make easier the comparison with previous literature we remark that the G_1 used here is equal to the invariant structure function W_1 defined in Eq. (2.13) and it is also equal to the invariant function of the same name used in [86], [87], [88] and [25]. This function can be easily extracted because is the coefficient of $-g^{\mu,\nu}$ in the hadronic tensor.

thus

$$G_1 = W_T \tag{3.24}$$

and

$$2G_1 = W^{\mu\nu}\left(-g_{\mu\nu}\right) \tag{3.25}$$

here we have used Eq.(2.18) and Table 3.2.

From the general relation (3.22) and assuming (3.25) we can obtain the following equations among them the Lam-Tung relation:

$$2G_1 = 3G_1 + G_2 + G_3 \tag{3.26}$$

and then, deduce an explicit relation among the G's

$$0 = G_1 + G_2 + G_3 \tag{3.27}$$

Using Eq.(3.22) and Eq.(2.21) we get

$$\begin{aligned} W_L &= G_1 + G_3 \\ -2W_{\Delta\Delta} &= G_2 \end{aligned}$$

Eq.(3.27) together with the above results allow us to find an equivalent relation in terms of the helicity structure functions,

$$W_L = 2W_{\Delta\Delta} \tag{3.28}$$

At the parton level this result is trivial since we have that both functions are equal to zero. The importance will be seen once we move to next-to-leading order predictions. Equations (3.25) and also (3.28) are known as the Lam-Tung relation [86] and are the analogues of the Callan-Gross relation in DIS [86]. The Lam-Tung relation is independent of the lepton c.m.s chosen and depends fundamentally on Eq.(3.25).

The Lam-Tung result can also be described in terms of the λ, μ, ν, defined in Eq.(2.26):

$$1 - \lambda - 2\nu = 0 \tag{3.29}$$

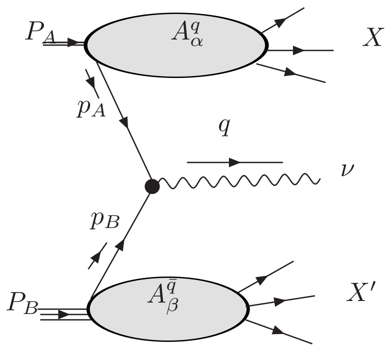
or in terms of the A's described in the last chapter, equations (2.27) and (2.28):

$$A_0 = A_2 \tag{3.30}$$

The Lam-Tung relation just expresses the fact that at high energies the dominant cross section is for the production of a virtual photon with transverse polarization. This is a direct consequence of collinear partons with spin 1/2 .

3.1.3 The parton model in quantum field theory

We want now to give a field theoretical explanation of the parton model. Thus we need to start with the Feynman diagram,



and the corresponding amplitude:

$$\mathcal{M} = \frac{1}{q^2}\bar{u}\left(p_A\right)\left(i\gamma^\nu\right)A^q\left(P_A, X\right)v\left(p_B\right)\left(i\gamma_\nu\right)A^{\bar{q}}\left(P_B, X'\right)$$

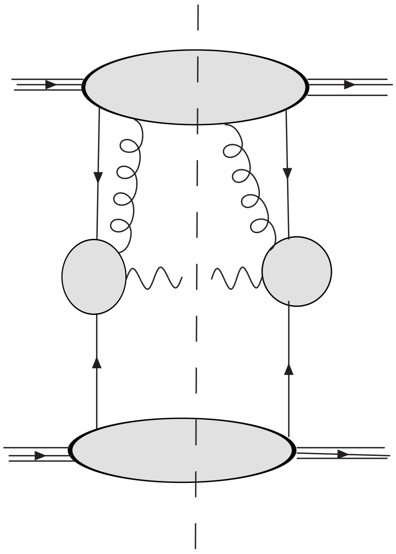
where,

$$\begin{aligned} A^q_\alpha\left(P_A, X\right) &\equiv \left\langle X\right|\psi_\alpha(0)\left|P_A\right\rangle \\ A^{\bar{q}}_\beta\left(P_B, X'\right) &\equiv \left\langle X'\right|\bar{\psi}_\beta(0)\left|P_B\right\rangle \end{aligned}$$

with $\psi_\alpha(x)$ and $\bar{\psi}_\alpha(x)$ the quark and antiquark fields (we are suppressing any other labels necessary to specify the state of the interacting hadrons). A^q and $A^{\bar{q}}$ can be considered as the amplitudes to produce a virtual quark or antiquark in the transitions $\mid P_A\rangle \rightarrow \mid X\rangle$ or $\mid P_B\rangle \rightarrow \mid X'\rangle$. In the above Feynman diagram we are going to make the following two assumptions:

1. The interaction is given by the Born approximation.
2. The soft matrix elements $|A|^2$ fall off rapidly for p_i off of the mass shell and for p_i non-collinear with the parent hadron momentum P_i, i.e the soft matrix elements only important when $p_i^2 \approx 0$ and $p_i^\mu P_{\mu\, i} \approx 0$

The first assumption means that diagrams like,



are not the dominant contributions to $W^{\mu\nu}$ in the limit $Q^2 \to \infty$, $S \to \infty$ with Q^2/S fixed.

We need now to introduce light-cone coordinates. They can be seen as a change of coordinates from the usual $(0,1,2,3)$ or (t,x,y,z) [49]. Given an arbitrary vector V^μ, we define

$$V^+ \equiv \frac{V^0 + V^3}{\sqrt{2}} \tag{3.31}$$

$$V^- \equiv \frac{V^0 - V^3}{\sqrt{2}} \tag{3.32}$$

$$V^T \equiv \vec{V}_T = (V_x, V_y) \tag{3.33}$$

with[7] $V^2 = 2V^-V^+ - (V^T)^2$. Thus we can write $V^\mu = \left(V^+, V^-, V^T\right)$. We will introduce also some "unit" vectors along the plus, minus and transverse directions:

$$\begin{aligned} n^+ &\equiv \left(1,0,0^T\right) \\ n^- &\equiv \left(0,1,0^T\right) \\ n^T &\equiv \left(0,0,\vec{1}\right) \end{aligned} \tag{3.34}$$

it is easy to see that,

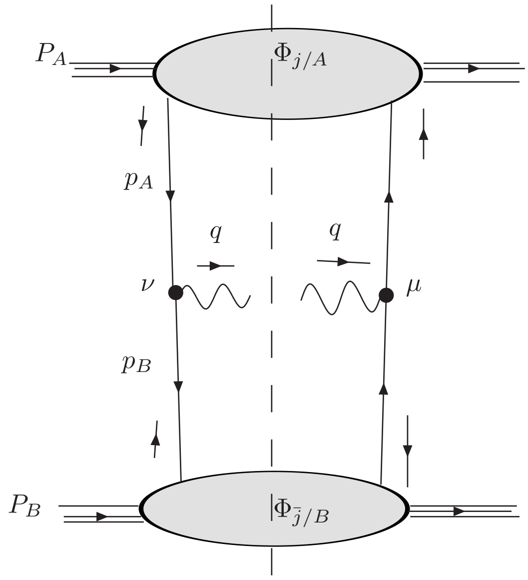
$$n^+ \cdot n^+ = n^- \cdot n^- = n^+ \cdot n^T = n^- \cdot n^T = 0 \tag{3.35}$$

[7] In general $A \cdot B = A^+B^- + A^-B^+ - A^T \cdot B^T$

and

$$n^+ \cdot n^- = 1 \tag{3.36}$$

Notice that the mathematical value of each Feynman diagram is dependent upon the particular gauge being used. With this in mind we will use the light-cone gauge[8] [51], [112] $A^+ = 0$ which can also be written in covariant way $n^-_\mu A^\mu(x) = 0$. It turns out that in the family of physical gauges, diagrams like the one above are not important [91]. So we can affirm safely that the hadron tensor is given at Born level by the diagram



that corresponds to the annihilation of a quark j from hadron A and antiquark $\bar{j}$ from hadron B plus a similar diagram with the antiquark coming from A and a quark from B. So the hadronic tensor is at lowest order in α_S:

$$W^{\mu\nu} = S \sum_j e_j^2 \int \frac{d^4 p_A}{(2\pi)^4} \int \frac{d^4 p_B}{(2\pi)^4} (2\pi)^4 \, \delta^4(p_A + p_B - q) Tr \left[\Phi_{j/A}(P_A; p_A) \, \gamma^\mu \Phi_{\bar{j}/B}(P_B; p_B) \, \gamma^\nu \right] \tag{3.37}$$

[8]It is also possible to use other physical gauges. For example we can change n^- to $\tilde{n} = \left(1/\sqrt{2}, 1/\sqrt{2}, 0_T\right)$ with $\tilde{n}^2 = -1$ [111], [42].

The sum in $W^{\mu\nu}$ is over all quark flavors and we have averaged over quark colors. Here, we need to define the quark and antiquark correlation matrices [9] [109],[115]:

$$\Phi_{j/A}\left(P_A;p_A\right)_{\alpha\beta} \equiv \int d^4y\, e^{-ip_A\cdot y}\left\langle P_A\left|\bar{\psi}^{(j)}_\beta(y)\psi^{(j)}_\alpha(0)\right| P_A\right\rangle \tag{3.38}$$

$$\Phi_{\bar{j}/B}\left(P_B;p_B\right)_{\alpha\beta} \equiv \int d^4y\, e^{-ip_B\cdot y}\left\langle P_B\left|\psi^{(j)}_\alpha(y)\bar{\psi}^{(j)}_\beta(0)\right| P_B\right\rangle \tag{3.39}$$

where $\psi^{(j)}$ and $\bar{\psi}^{(j)}$ are respectively the unrenormalized quark and antiquark field operators of flavor j.

Let us check the consequences of assumption 2. We have

$$p^2 = 2p^+p^- - \left(p^T\right)^2 \approx 0$$

$$p^\mu P_\mu = P^+p^- + P^-p^+ - P^T\cdot p^T \approx 0$$

In terms of the light-cone coordinates the beam momentum is:

$$P_A = \left(\sqrt{\frac{S}{2}}, 0, 0_T\right) \tag{3.40}$$

and the target momentum:

$$P_B = \left(0, \sqrt{\frac{S}{2}}, 0_T\right) \tag{3.41}$$

thus

$$p_A\cdot P_A = P_A^+p_A^- \approx 0$$

$$p_B\cdot P_B = P_B^-p_B^+ \approx 0$$

it is easy to see that $p_A^T \approx 0$, $p_B^T \approx 0$, $p_A^- \approx 0$ and $p_B^+ \approx 0$. Therefore, $\Phi_{j/A}$ will fall off very quickly when p_A^T and p_A^- get large and in the same way $\Phi_{\bar{j}/B}$ will not contribute when p_B^T and p_B^+ are large. We will ignore the quark transverse motion and only the collinear configuration will be considered. Setting

$$p_A^+ = \xi_A P_A^+ \tag{3.42}$$

and

$$p_B^- = \xi_B P_B^- \tag{3.43}$$

[9] Averages over color and spin are understood in this definition.

the delta function now is:

$$\begin{aligned}
\delta^4(p_A+p_B-q) &= \delta(p_A^+ + p_B^+ - Q^+)\delta(p_A^- + p_B^- - Q^-)\delta^2(\vec{Q}_T) \\
&= \delta(p_A^+ - \xi_A P_A^+)\delta(p_B^- - \xi_B P_B^-)\delta^2(\vec{Q}_T) \\
&= \left[\frac{1}{P_A^+}\delta\left(\xi_A - \frac{p_A^+}{P_A^+}\right)\right]\left[\frac{1}{P_B^-}\delta\left(\xi_B - \frac{p_B^-}{P_B^-}\right)\right]\delta^2(\vec{Q}_T)
\end{aligned}$$

where $q = \xi_A P_A^+ + \xi_B P_B^-$. Using the above result and

$$\begin{aligned}
d^4p_A &= P_A^+ d\xi_A dp^- dp^T \\
d^4p_B &= P_B^- d\xi_B dp^+ dp^T
\end{aligned}$$

we can write the hadronic tensor (3.37) as:

$$W^{\mu\nu} = S\delta^2(\vec{Q}_T)(2\pi)^2 \sum_j e_j^2\, Tr\left[\left(\mathcal{F}_{j/A}\right)\gamma^\mu\left(\mathcal{F}_{\bar{j}/B}\right)\gamma^\nu\right] \tag{3.44}$$

where we have defined,

$$\begin{aligned}
\left(\mathcal{F}_{j/A}\right)_{\alpha\beta} &\equiv \int \frac{dp_A^T}{(2\pi)^2}\frac{dp_A^-}{2\pi} d\xi_A\, \delta\left(\xi_A - \frac{p_A^+}{P_A^+}\right)\left(\Phi_{j/A}\right)_{\alpha\beta} \\
\left(\mathcal{F}_{\bar{j}/B}\right)_{\alpha\beta} &\equiv \int \frac{dp_B^T}{(2\pi)^2}\frac{dp_B^+}{2\pi} d\xi_B\, \delta\left(\xi_B - \frac{p_B^-}{P_B^-}\right)\left(\Phi_{\bar{j}/A}\right)_{\alpha\beta}
\end{aligned}$$

Note that

$$\left(\mathcal{F}_{j/A}\right)_{\alpha\beta} = \int d\xi_A\, \delta\left(\xi_A - \frac{p_A^+}{P_A^+}\right)\int dy^-\, e^{-ip_A^+ y^-}\left\langle P_A \left|\bar{\psi}_\beta^{(j)}(0, y^-, 0_T)\psi_\alpha^{(j)}(0)\right| P_A\right\rangle \tag{3.45}$$

and

$$\left(\mathcal{F}_{\bar{j}/B}\right)_{\alpha\beta} = \int d\xi_B\, \delta\left(\xi_B - \frac{p_B^-}{P_B^-}\right)\int dy^+\, e^{-ip_B^- y^+}\left\langle P_B \left|\psi_\alpha^{(j)}(y^+, 0, 0_T)\bar{\psi}_\beta^{(j)}(0)\right| P_B\right\rangle \tag{3.46}$$

We want now to parametrize $\left(\mathcal{F}_{j/A}\right)$ and $\left(\mathcal{F}_{\bar{j}/B}\right)$. Since they are 4×4 matrices we can use a general decomposition in a basis of Dirac matrices. For example we can use $\left\{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5, \gamma^5\gamma^\mu\right\}$ to write

$$\left(\mathcal{F}_{j/A}\right)_{\alpha\beta} = \frac{1}{2}\left\{\mathscr{S}\mathbb{1} + \mathscr{V}_\mu\gamma^\mu + \mathscr{A}_\mu\gamma^5\gamma^\mu + \mathscr{P}\gamma^5 + \mathscr{T}_{\mu\nu}\sigma^{\mu\nu}\right\}$$

where the factor 1/2 is introduced for later convenience. The quantities $\mathscr{S}, \mathscr{V}_\mu, \mathscr{A}_\mu, \mathscr{P}, \mathscr{T}_{\mu\nu}$ are only functions of the vectors P_A^μ and p_A^μ. Borrowing from DIS the fact that the partonic part is always odd in gamma matrices, if we neglect the quark mass, we conclude that only $\left\{\gamma^\mu, \gamma^5\gamma^\mu\right\}$ can contribute. For spin averaged process $\gamma^5\gamma^\mu$ is not present so we are left with γ^μ. So $\left(\mathcal{F}_{j/A}\right)_{\alpha\beta}$ is equal to:

$$\left(\mathcal{F}_{j/A}\right)_{\alpha\beta} = \frac{1}{2}\mathscr{V}_\mu \left(\gamma^\mu\right)_{\alpha\beta} \tag{3.47}$$

where $\mathscr{V}^\mu$ is given by:

$$\mathscr{V}^\mu = Tr\left[\frac{1}{2}\left(\mathcal{F}_{j/A}\right)\gamma^\mu\right] \tag{3.48}$$

Inserting this identity in (3.45) we obtain:

$$\mathscr{V}^\mu = \frac{1}{2}\int d\xi_A\,\delta\left(\xi_A - \frac{p_A^+}{P_A^+}\right)\int dy^-\, e^{-ip_A^+ y^-}\left\langle P_A \left| \bar{\psi}_\beta^{(j)}(y^+ = 0, y^-, y^T = 0)\gamma^\mu_{\alpha\beta}\psi_\alpha^{(j)}(0)\right| P_A\right\rangle \tag{3.49}$$

Now $\mathscr{V}^\mu$ can only be function of the vector P_A^μ and p_A^μ thus

$$\mathscr{V}^\mu\left(P_A, \xi_A\right) = \mathscr{V}_{j/A}\left(P_A, \xi_A\right)P_A^\mu + \mathcal{O}(n^\mu) \tag{3.50}$$

with

$$\mathscr{V}_{j/A} = \frac{\mathscr{V}^\mu n_\mu^-}{P_A^+} \tag{3.51}$$

where n_μ^- is the unit vector along the minus direction defined above and the corrections depending on this vector are power suppressed. So, we have:

$$\mathscr{V}_{j/A} = \frac{1}{2P_A^+}\int d\xi_A\,\delta\left(\xi_A - \frac{p_A^+}{P_A^+}\right)\int dy^-\, e^{-ip_A^+ y^-}\left\langle P_A \left| \bar{\psi}^{(j)}(y^+ = 0, y^-, y^T = 0)\gamma^+\psi^{(j)}(0)\right| P_A\right\rangle \tag{3.52}$$

and finally putting all together

$$\mathcal{F}_{j/A} = \frac{1}{2P_A^+}\int d\xi_A\,\delta\left(\xi_A - \frac{p_A^+}{P_A^+}\right)\int dy^-\, e^{-ip_A^+ y^-}\left\langle P_A \left| \bar{\psi}^{(j)}(y^+ = 0, y^-, y^T = 0)\gamma^+\psi^{(j)}(0)\right| P_A\right\rangle\left(\frac{\not{P}_A}{2}\right) \tag{3.53}$$

here $\not{P}_A = P_A^\mu\gamma_\mu$.

We can pause for a second to compare equations (3.45) and (3.53). In the last one we have separated the spinor and Lorentz indices which are now concealed in $\not{P}_A/2$ and we are left with a scalar function that contains all the pertinent information.

Substituting (3.53) into (3.44) we find:

$$\begin{aligned}
W^{\mu\nu} = \;& S \quad \delta^2(\vec{Q}_T)(2\pi)^2 \sum_j e_j^2 \left(\frac{1}{2}\right)^2 Tr\left[\not{P}_A \gamma^\mu \not{P}_B \gamma^\nu\right] \\
\times \;& \frac{1}{2P_A^+} \int d\xi_A\, \delta\left(\xi_A - \frac{p_A^+}{P_A^+}\right) \int dy^-\, e^{-ip_A^+ y^-} \left\langle P_A \left| \bar{\psi}^{(j)}(0, y^-, 0_T)\gamma^+ \psi^{(j)}(0) \right| P_A \right\rangle \\
\times \;& \frac{1}{2P_B^-} \int d\xi_B\, \delta\left(\xi_B - \frac{p_B^-}{P_B^-}\right) \int dy^+\, e^{-ip_B^- y^+} \left\langle P_B \left| Tr\left\{\gamma^- \psi^{(j)}(y^+, 0, 0_T)\bar{\psi}^{(j)}(0)\right\} \right| P_B \right\rangle
\end{aligned} \tag{3.54}$$

which can be rewritten as

$$\begin{aligned}
W^{\mu\nu} = \;& S \quad \delta^2(\vec{Q}_T)(2\pi)^4 \sum_j e_j^2 \\
\times \;& \int d\xi_A \delta\left(\xi_A P_A^+ - p_A^+\right) f_{j/A}\left(\xi_A\right) \\
\times \;& \int d\xi_B \delta\left(\xi_B P_B^- - p_B^-\right) f_{\bar{j}/B}\left(\xi_B\right) \\
\times \;& \frac{1}{4} Tr\left[\not{p}_A \gamma^\mu \not{p}_B \gamma^\nu\right]
\end{aligned} \tag{3.55}$$

where we have defined after comparing with (3.3) [42], [51]:

$$f^0_{j/A}\left(\xi_A\right) \equiv \frac{1}{4\pi} \int dy^-\, e^{-i\xi_A P_A^+ y^-} \left\langle P_A \left| \bar{\psi}^{(j)}(0, y^-, 0_T)\gamma^+ \psi^{(j)}(0) \right| P_A \right\rangle_{A^+=0} \tag{3.56}$$

$$f^0_{\bar{j}/B}\left(\xi_B\right) \equiv \frac{1}{4\pi} \int dy^+\, e^{-i\xi_B P_B^- y^+} \left\langle P_B \left| Tr\left\{\gamma^- \psi^{(j)}(y^+, 0, 0_T)\bar{\psi}^{(j)}(0)\right\} \right| P_B \right\rangle_{A^-=0} \tag{3.57}$$

The reader should not get confused for the apparent differences between the definitions (3.56) and (3.57). As it is denoted, they are defined in different gauges and they need to be renormalized. To obtain a gauge invariant definition we follow the standard procedure to introduce a *Wilson line* between the quark and antiquark fields [51], [102], [112]:

$$f^0_{j/A}\left(\xi_A\right) = \frac{1}{4\pi} \int dy^-\, e^{-i\xi_A P_A^+ y^-} \left\langle P_A \left| \bar{\psi}^{(j)}(0, y^-, 0_T)\gamma^+ \mathcal{O}_0 \psi^{(j)}(0) \right| P_A \right\rangle$$

where

$$\mathcal{O}_0 = \mathcal{P}\exp\left(ig_0\int_0^{y-} dz^- A^+_{0,a}\left(0,z^-,0_T\right)t_a\right)$$

Here $\mathcal{P}$ denotes a path-ordered product, while the t_a are the generators for the **3** representation of $SU(3)$. There is also an implied sum over the color index a.

To see why we need renormalization we can for example rewrite (3.56),

$$f^0_{j/A}\left(\xi_A\right) = \sum_N \delta\left(P^+_A(1-\xi_A) - P^+_N\right)\frac{1}{2}\left\langle P_A\left|\bar{\psi}(0)\gamma^+\left|P_N\right\rangle\left\langle P_N\right|\psi(0)\right|P_A\right\rangle \tag{3.58}$$

so it is clear that these matrix elements are UV divergent since they contain outgoing states $|P_N\rangle$ with unbounded transverse and minus momenta [51], [112], [115]. The presence of UV divergences does not allow us to interpret the above distributions as number densities [112]. The renormalization can be done by ordinary UV renormalization of the field operators [51], for example the MS [42], $\overline{\mathrm{MS}}$ [46], [112] or DIS [30] schemes can be used. So we finally arrive to:

$$\begin{aligned} f_{j/A}\left(\xi,\mu_F\right) &= \frac{1}{4\pi}\int dy^-\, e^{-i\xi P^+y^-}\left\langle P_A\left|\bar{\psi}^{(j)}(0,y^-,0_T)\gamma^+\mathcal{O}\psi^{(j)}(0)\right|P_A\right\rangle_R && (3.59)\\ f_{\bar{j}/A}\left(\xi,\mu_F\right) &= \frac{1}{4\pi}\int dy^-\, e^{-i\xi P^+y^-}\left\langle P_A\left|Tr\left\{\gamma^+\psi^{(j)}(0,y^-,0_T)\hat{\mathcal{O}}\bar{\psi}^{(j)}(0)\right\}\right|P_A\right\rangle_R && (3.60)\end{aligned}$$

here

$$\begin{aligned}\mathcal{O} &= \mathcal{P}\exp\left(ig\int_0^{y-} dz^- A^+_a\left(0,z^-,0_T\right)t_a\right)\\ \hat{\mathcal{O}} &= \mathcal{P}\exp\left(-ig\int_0^{y-} dz^- A^+_a\left(0,z^-,0_T\right)t^T_a\right)\end{aligned}$$

Note that with the renormalization a scale μ_F is introduced. The evolution of the parton distributions with the scale μ_F is given by the DGLAP equations [42], [112] :

$$\mu_F^2\frac{d}{d\mu_F^2}f_{j/A}\left(x,\mu_F\right) = \int_x^1\frac{d\xi}{\xi}\sum_k P_{j/k}\left(\frac{x}{\xi},\alpha_S(\mu_F)\right)f_{j/A}\left(\xi,\mu_F\right) \tag{3.61}$$

with $P_{j/k}$ the Altarelli-Parisi kernel expanded in α_S to a suitable order and k runs over quark and antiquark flavors and the gluon.

The definitions (3.59) and (3.60) are related by charge conjugation. We can take Eq. (3.59) as antiquark distribution if we define [115]

$$\gamma^+\psi^{(j)} \equiv \bar{\psi}^{(\bar{j})}, \quad \bar{\psi}^{(j)} \equiv \psi^{(\bar{j})} \tag{3.62}$$

For completeness we include here the definition of the gluon distribution [42], [112] :

$$f_{g/A}(\xi,\mu_F) = \frac{1}{2\pi\xi P_A}\int dy^- \; e^{-i\xi P^+ y^-} \left\langle P_A \left| F_a^+(0,y^-,0_T)\mathcal{O}_{ab}F_b^+(0)\right| P_A\right\rangle_R \tag{3.63}$$

where

$$\mathcal{O}_{ab} = \mathcal{P}\exp\left(ig\int_0^{y-} dz^- A_c^+\left(0,z^-,0_T\right)t_c\right)$$

Here t_c are the generators of the **8** dimensional representation of $SU(3)$.

3.2 QCD Picture

Quantum Chromodynamics (QCD) is the non-Abelian gauge field theory that describes the strong interaction. In order to understand how QCD generalizes the parton model and modifies the naive DY picture previously described we will make a short detour in order to give a part of the historical background.

By the end of the 1960's and beginning of the 1970's several experimental facts were present that made the theoretical picture confusing. From the 1950's until today we have a continuously increasing set of particles, hadrons, [61] that behave in ways that are reminiscent of the proton and neutron so very early it was postulated that all of them were composed by "smaller" more fundamental entities , called quarks, but despite intensive searches free quarks were not seen. So they became a useful mathematical fiction [124]. But after the DIS experiments of 1969 [18], the discovery of asymptotic freedom [74], [75] and the DY experiments at BNL [34] their physical reality was accepted and they are now among the fundamental constituents of matter [61]. Quarks exhibit remarkable properties: their electric charges are fractions $\left(\frac{1}{3} \text{ or } \frac{2}{3}\right)$ of the charge of the electron, they only appear in sets of two or three and when we want to break them apart it is easier to obtain again sets of two or three than to isolate one. So they interact quite strongly.

The attentive reader should be asking now: how is that possible? The parton model picture described just two sections ago, shows hadrons as bags of quarks behaving as almost free point-like particles and now we cannot take them apart.

To solve this apparent paradox we will follow the ideas of David Gross [76]. In 1968 Callan and Gross [31] discovered, using current algebras, an interesting sum rule for the structure functions F_1 and F_2 of DIS:

$$2xF_1(x) = F_2(x) \tag{3.64}$$

where x is the Bjorken variable. It was precisely Bjorken whom in fall of the same year noted that this sum rule together with dimensional analysis would suggest scaling in DIS [76]. This scaling is of the same type described in Eq.(3.16) and was observed for first time at the DIS experiment at SLAC of 1969 [18]. Relation (3.64) also implies [32]:

$$\frac{\sigma_L}{\sigma_T} \to 0 \tag{3.65}$$

when $Q^2 \to \infty$ and where σ_L (σ_T) is the cross section for the scattering of longitudinal (transverse) polarized photons. This equation made possible to determine the spin of the constituents of the nucleons, since $\sigma_L = 0$ is the case for particles of spin 1/2, while at the same time $\sigma_T = 0$ is the case for scalar particles. As has already been said before this is equivalent to Eq.(3.28) in the DY process.

By 1969 Gross was convinced that [76]

> *"...in a field-theoretic context only a free, noninteracting theory could produce exact scaling"*

So, he set to prove that no gauge theory could have such behavior. In modern terms his research plan was to prove that there was no gauge theory with asymptotic freedom, of course he proved himself wrong, but his instinct about the importance of scaling was the crucial insight.

3.2.1 Asymptotic Freedom

The explanation of scaling is the essential characteristic of QCD and paved the way to make this theory "the theory" of strong interactions. Asymptotic freedom is the term to describe the decrease of α_S, the "strong coupling constant," at short distances and its increase toward longer distances and times. Asymptotic freedom elucidates how the quarks can behave almost-freely, a requirement from scaling, and its flip side that the coupling increases with distance, a phenomenon known as confinement [10]. Analytically, we can prove asymptotic freedom by calculating the dependence of the coupling constant from the renormalization scale μ [74], [75], [103]. This is done solving the renormalization group equation [11]:

$$\mu \frac{d}{d\mu} \frac{\alpha_S(\mu)}{\pi} = -\beta_0 \left(\frac{\alpha(\mu)}{\pi} \right)^2 - \beta_1 \left(\frac{\alpha(\mu)}{\pi} \right)^3 - \mathcal{O}(\alpha_S^4)$$

This derivative can be calculated pertubatively in QCD. The fist two coefficients are known [30], [48]:

$$\begin{aligned} \beta_0 &= \frac{33 - 2n_f}{12} \\ \beta_1 &= \frac{306 - 38n_f}{48} \end{aligned}$$

where n_f is the number of quark flavors. To find an approximate solution we can set all β_i, with $i \geq 1$, equal to zero to obtain:

$$\mu \frac{d}{d\mu} \frac{\alpha_S(\mu)}{\pi} = -\beta_0 \left(\frac{\alpha(\mu)}{\pi} \right)^2$$

and solving for $\alpha_S(\mu)$ we find

$$\alpha_S(\mu) = \frac{\alpha_S(\mu_0)}{1 + \alpha_S(\mu_0) \frac{\beta_0}{\pi} \ln \left(\frac{\mu^2}{\mu_0^2} \right)} \tag{3.66}$$

Here we have used $\alpha_S(\mu)|_{\mu_0} = \alpha_S(\mu_0)$ as boundary condition. We can choose, for instance, $\alpha_S (\mu_0 \approx M_Z) \approx 0.112$ with $M_Z \approx 91 GeV$. It is easy to see how the sign of β_0 defines the

[10] Confinement remains to be analytically proved, see the million dollar price at [36]

[11] This particular example is performed in the $\overline{\text{MS}}$ scheme

behavior of the coupling constant. Since $\beta_0 > 0$ in QCD, we have $\alpha_S(\mu \to \infty) \to 0$. With asymptotic freedom the strong interaction at high energies becomes "weak" and so perturbative methods are useful. Thus, asymptotic freedom can be applied to observables that are dominated by the short-distance, high energy behavior of QCD.

Asymptotic freedom can be interpreted as the antiscreening or strengthening of applied magnetic fields in paramagnetic materials (those materials whose magnetic moments align with an applied field). This behavior is a consequence of the self-coupling of the gluons and produces the 33/12 term in β_0, while the quarks produce the competing effect of screening; thus the $-2n_f/12$ is analogous to screening in diamagnetic materials (those whose internal magnetic field opposed the applied field) [118].

3.2.2 The choice of scale

[12]The μ present in the previous section is an arbitrary scale introduced during renormalization and determines the strength of the interaction. In principle, it can have any finite value. In standard perturbative QCD, pQCD, we can expand, for example, a cross section in the following way[13]:

$$\sigma\left(\frac{Q_i^2}{\mu^2}, \frac{Q_j^2}{Q_j^2}, \frac{p_i^2}{\mu^2}, \frac{m^2}{\mu^2}, \alpha_S(\mu)\right) = \sum_{n=1}^{\infty} C_n\left(\frac{Q_i^2}{\mu^2}, \frac{Q_j^2}{Q_i^2}, \frac{p_i^2}{\mu^2}, \frac{m^2}{\mu^2}\right)\left(\frac{\alpha_S(\mu)}{\pi}\right)^n \tag{3.67}$$

where Q_i^2 and Q_j^2 are large external momenta which define the energy exchange of the process, p_i^2 represents the small external invariants, like small masses of observed external particles, m^2 is the mass scale of the colliding partons: quarks and gluons and μ is the renormalization scale or factorization scale. We will neglect the small mass scales p_i^2 and m_i^2 to fix the value of α_S, since we want it as small as we can. Very often the coefficient functions C_n depend logarithmically on the ratios of all the mass scales shown in Eq.(3.67); then, if we select a value μ very different from the large scales Q_i we will have large logarithms and as consequence we

[12]This section and the next one follow closely [105]

[13]Any physical observable is independent of the renormalization scale

can spoil the perturbative expansion. Choosing $\mu^2 \approx Q_i^2$ we encounter:

$$\frac{Q_i^2}{Q_j^2} \equiv \chi_{ij} \approx \mathcal{O}(1)$$

but

$$\frac{p_i^2}{Q_i^2} \approx \mathcal{O}(0), \quad \frac{m_i^2}{Q_i^2} \approx \mathcal{O}(0),$$

here we have assumed that the quark masses and the hadron masses are small compared with the large scale of the process. Logarithms of the above quantities are quite large and thus terms like $\alpha_S\left(\mu \approx Q_i\right) \times \ln\left(m^2/Q_i^2\right)$ are not small making the expansion unusable. The sensitivity to the masses of the partons is known as *infrared sensitivity.* Any observable with such dependence cannot be reliably calculated in pQCD.

The smart reader should be arguing now that the quarks and gluons are not observed in the detectors so the dependence maybe is just for the masses of the hadrons if they are observed. Generally, such masses are small compared with the large exchange scale masses of the process. So, we are back to the same problem. We can conclude that pQCD can be used for observables which are not sensitive to the masses of the partons or hadrons involved. Such physical quantities are known as *infrared safe*, IR safe. For this type of quantities we can safely select $\mu \approx Q$.

Quantitatively, infrared safe observables have the following behavior [115]:

$$\lim_{\mu\to\infty} F\left(\frac{Q_i^2}{\mu^2}, \frac{p_i^2}{\mu^2}, \frac{m(\mu)^2}{\mu^2}, \alpha_S(\mu)\right) = f\left(\frac{Q_i^2}{\mu^2}, \alpha_S(\mu)\right) + \mathcal{O}\left(\left(\frac{m^2}{\mu^2}\right)^a\right), \quad a > 0$$

which means that F should approach a limit as $\frac{m}{\mu} \to 0$ with $\frac{Q}{\mu}$ fixed with corrections that vanish as a positive power of $\frac{m}{\mu}$. The above equation just tells us that the larger the momentum scale in the process, the smaller α_S is and the better the perturbative expansion will be.

Going back to the cross section (3.67), we can write for $\mu \approx Q$

$$\sigma\left(\frac{Q_i^2}{\mu^2}, \chi_{ij}, \alpha_S(\mu)\right) = \sigma\left(1, \chi_{ij}, \alpha_S(\mu)\right) = \sum_{n=1}^{\infty} C_n\left(1, \chi_{ij}, \right)\left(\frac{\alpha_S(\mu)}{\pi}\right)^n$$

Since the χ_{ij} are of order 1, we can conclude that pQCD works well when we have a single big scale or several scales of the same size. If we want to find the cross section for the DY process at given Q_T two or more physical scales of different sizes are present[14], thus $\chi_{ij} \approx \frac{Q^2}{Q_T^2}$ which generates large logarithms in the coefficient functions. In consequence, pQCD does not work well in this or similar cases and it is necessary, in order to obtain sensible results, to resum these large logarithms to all orders in α_S.

The DY process and other collisions like DIS are not fully IR safe. This can also be seen from the observation that cross sections involving hadrons in the initial state should be sensitive to the mass scale of the hadrons involved. The solution here is to "separate" the short distance physics from the long distance, long-time scales included in the collision. Factorization theorems are precisely the recipes to perform such separation in pQCD. The long-distance physics is factorized in non-perturbative, well defined and universal functions that can be measured in some experiments and used in others.

We can conclude this section reviewing what quantities can be calculated in pQCD:

- Infrared safe cross sections, like $\sigma^{total}\,(e^+e^- \to hadrons)$ and jet cross sections.
- Factorizable cross sections like DIS and DY where the IR-dependence can be factorized in universal functions:

$$(Observable)[Q^2] = (IR_{Safe})\left[\frac{Q^2}{\mu_F^2}\right] \otimes (IR_{Sensitive})\left[\mu_F^2\right]_{Universal}$$

- Q^2-dependence of factorizable cross sections. Despite the fact that pQCD cannot calculate the absolute value of the factorizable cross sections, the Q^2-dependence is within what is possible in pQCD because the dependence is defined by what happens around the big scale Q^2. We can calculate the Q^2-evolution because renormalization-group in-

[14]The bulk of the data for observed transverse momentum Q_T is for the region with $Q_T^2 \ll Q^2$ where Q^2 is the invariant mass of the dilepton.

variants, F, obey the equation

$$\left(\mu\frac{d}{d\mu}-\Gamma\right)F=0$$

with Γ an anomalous dimension.

3.2.3 Infrared safety for processes with initial hadrons

As we will see in the next section, the cross section for hadron-hadron collisions can be written as:

$$\begin{aligned} d\sigma(\Gamma) &= \sum_{n=2}^{\infty}\sum_{a,b}\int d\xi_A\int d\xi_B f_{a/A}\left(\xi_A,\mu_F\right)f_{b/B}\left(\xi_B,\mu_F\right)\\ &\times \int dy_1\int dQ_{T1}\int dy_2\int dQ_{T2}\ldots\int dy_n\int dQ_{Tn}\\ &\times \frac{d\hat{\sigma}^{(n)}}{dy_1 dQ_{T1}dy_2 dQ_{T2}\ldots dy_n dQ_{Tn}}\Gamma_n\left(k_1^\mu,k_2^\mu,\ldots,k_n^\mu\right)\end{aligned}$$

where y_n and Q_{Tn} are the rapidity and transverse momentum of the n^{th} particle, $\Gamma_n\left(k_1^\mu,k_2^\mu,\ldots,k_n^\mu\right)$ are constraint functions invariant under the interchange of the n-particles and k_n^μ are the particle momenta. Different constraint functions correspond to different observables. For an IR-safe quantity we need [84] (1):

$$\Gamma_{n+1}\left(k_1^\mu,k_2^\mu,\ldots,(1-\lambda)k_n^\mu,k_n^\mu\right)=\Gamma_n\left(k_1^\mu,k_2^\mu,\ldots,k_n^\mu\right)$$

with $0\leq\lambda\leq 1$. This equation means that the constraint functions do not distinguish between states in which one set of collinear particles is substituted for another set with the same total momentum or when zero momentum particles are absorbed or emitted [116]. (2) we also need [105]:

$$\begin{aligned}\Gamma_{n+1}\left(k_1^\mu,k_2^\mu,\ldots,k_n^\mu,\lambda P_A^\mu\right) &= \Gamma_{n+1}\left(k_1^\mu,k_2^\mu,\ldots,k_n^\mu,\lambda P_B^\mu\right)\\ &= \Gamma_n\left(k_1^\mu,k_2^\mu,\ldots,k_n^\mu\right)\end{aligned}$$

where once again $0\leq\lambda\leq 1$. This condition just requires the observable to be blind to the details in the regions parallel to either P_A or P_B. (3) An IR-safe observable also demands we remove any dependence from the region parallel to both hadrons [105]:

$$d\hat{\sigma}=d\sigma-\text{initial state collinear counter-terms}$$

3.2.4 Factorization theorem for Drell-Yan

The field theory realization of the parton model is the factorization of long-distance from short-distance. As we have already hinted factorization theorems require [30], [46]:

1. All Lorentz invariants defining the process are large and comparable except for particle masses,
2. One counts all final states that include the specified outgoing particles or jets

The first condition means in the case of DY that S the square of the total center-of-mass energy and q^μ the momentum of the virtual photon γ^* are large with Q^2/S fixed. The transverse momentum Q_T of γ^* is either of order Q or is integrated over. The second condition means that we will consider the Drell-Yan process as $hadron_A + hadron_B \to \gamma^* + X$ where X denotes "anything else." For the situation of large measured Q_T the theorem says [3], [4], [30],[44], [62], [92], [93], [97] :

Factorization Theorem. *The sum of all diagrammatic contributions to the cross section is a direct generalization of the parton model result (3.15) and is equal to*

$$\begin{aligned} \frac{d\sigma_{h_A+h_B\to l^+l^-}}{dQ^2dyd^2\vec{Q}_Td\Omega} &= \tfrac{\alpha^2}{12SQ^2} \sum_{a,b}\int_{x_A}^1 \frac{d\xi_A}{\xi_A}\int_{x_B}^1\frac{d\xi_B}{\xi_B} f_{a/A}\left(\xi_A,\mu_F,\alpha_S(\mu)\right) f_{b/B}\left(\xi_B,\mu_F,\alpha_S(\mu)\right) \\ &\times \quad T_{ab}\left(Q_T,Q,\theta,\phi,\frac{x_A}{\xi_A},\frac{x_B}{\xi_B};\frac{Q^2}{\mu_F^2},\frac{\mu_F^2}{\mu^2},\alpha_S(\mu)\right) \end{aligned} \tag{3.68}$$

where the a, b sum is over all partons: quarks, antiquarks and gluons.

The hard scattering function

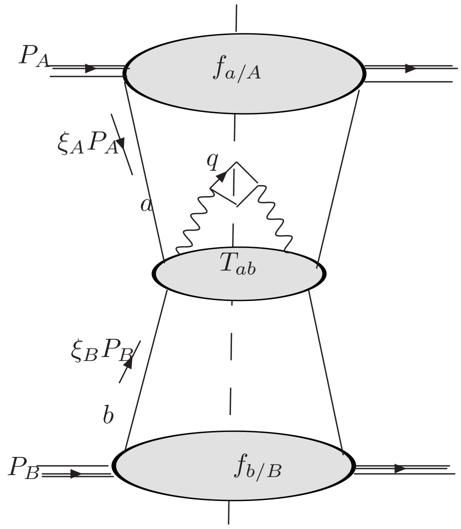
$$T_{ab}\left(Q_T,Q,\theta,\phi,\frac{x_A}{\xi_A},\frac{x_B}{\xi_B};\frac{Q^2}{\mu_F^2},\frac{\mu_F^2}{\mu^2},\alpha_S(\mu)\right)$$

is ultraviolet dominated and so computable in perturbation theory. It depends on the partons a, b, on the virtual photon γ^* and on the renormalization and factorization scales. But notice that is independent of the long-distance physics, so it is independent of the physics of the hadrons A and B. In contrast, the parton distributions

$$f_{a/A}\left(\xi_A,\mu_F,\mu\right), \quad f_{b/B}\left(\xi_B,\mu_F,\mu\right)$$

are IR dominated and are determined by the particular hadron involved in the collision. They also depend on μ_F. Since the parton distributions are independent of the particular hard scattering, they are universal and on this fact relies in great part the predicting power of this formalism.

Graphically we can illustrate the factorization theorem [30],



Compared with the formula from the parton model (3.15), we have now dependence from two mass scales: μ the renormalization scale and μ_F the factorization scale. The renormalization scale appears in any perturbative calculation, but the factorization scale is proper of observables where factorization is applied. μ_F defines the separation between the short-distance physics from the long-distance effects. Informally speaking, when calculating a diagram and integrating over k_T, the parton transverse momentum, one counts a contribution from $k_T^2 \leq \mu_F^2$ as part of $f(\xi, \mu)$, and from $k_T^2 > \mu_F^2$ as contribution to T_{ab}. The factorization scale appears in a way that is similar to the renormalization scale[15]. In actual calculations the separation is done in dimensional regularization, so things are more subtle that dividing an integral in two parts [113].

[15] See discussion before Eq.(3.59) and Eq.(3.60)

In calculations is often used $\mu = \mu_F$ and for DY is commonly chosen $\mu = \mu_F = Q$, selection that we will follow here, but this is not the only possibility and as with the renormalization scale, any physical observable should be independent of the particular choice of μ_F.

We would like to finish this section quoting George Sterman [118] and his insightful explanation about the factorization theorem,

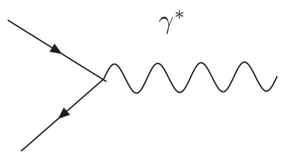
> *"...In the parton model, $f_{a/A}(x)$ denotes the density of partons a with momentum fraction x, a distribution that is assumed to be quantum-mechanically independent of the hard scattering at momentum transfer Q, and hence may be treated as an independent probability. In QCD, $f_{a/A}(x, \mu)$ represents the same density, but only of partons with transverse momentum $Q_T < \mu_F$. It is only these partons whose production may be considered incoherent with the hard scattering.*
> *If there were a maximum transverse momentum Q_{Tmax} for partons in the nucleon, $f_{a/A}(\xi_A, Q_{Tmax})$ would freeze for $\mu > Q_T$, and the theory would revert to the parton model above that scale. This is never the case, however, in a renormalizable field theory, and scale breaking measures the change in the density as the maximum transverse momentum increases. Of course, the structure functions and cross sections that we compute still depend on our choice of μ through uncomputed higher orders in T and evolution. "*

3.3 QCD corrections to Drell-Yan

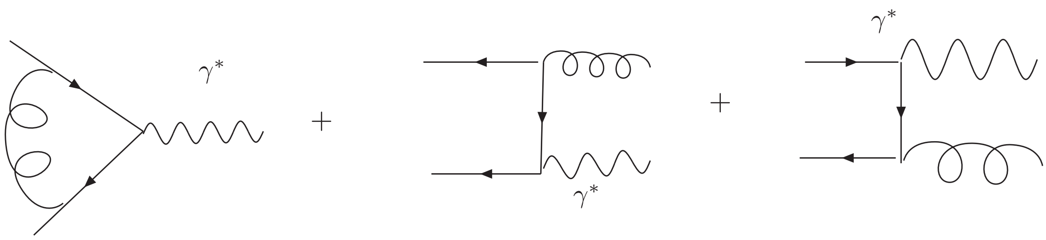
In order to calculate the next-to-leading order contributions to DY we need to find the $\mathcal{O}(\alpha_S)$ coefficient of the function T_{ab} defined in the Factorization Theorem. So let us first assume that[16] $\Lambda_{QCD} \ll Q \approx Q_T$ and consider, following the procedure outlined in Appendix B, the next set of diagrams:

Lowest order contribution,

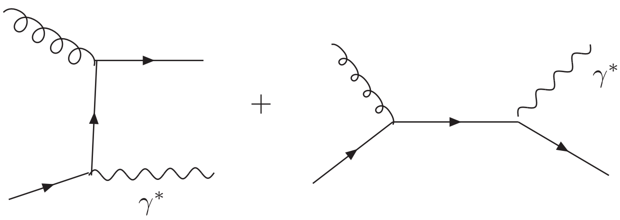
[16]The inequality guarantees that pQCD can be used and the equality makes possible the use of the Factorization Theorem.



gluon "radiation",



and contributions from the gluon content of the hadrons,



We are going to divide the terms from gluon radiation in two parts: *virtual corrections* and *real emission subprocess*. We will leave the virtual corrections for the next chapter since they only contribute when $Q_T = 0$. Thus the hadronic tensor for nonzero finite measured transverse momentum is given by:

$$W^{\mu\nu}_{NLO} = W^{\mu\nu(R)}_{j\bar{j}} + W^{\mu\nu}_{jg} + W^{\mu\nu}_{gj} \tag{3.69}$$

where

$$W^{\mu\nu}_{(\,)} = S \sum_j \int_{x_A}^1 \frac{d\xi_A}{\xi_A} \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{(\,)/A}\left(\xi_A;\mu\right) f_{(\,)/B}\left(\xi_B;\mu\right) w^{\mu\nu}_{(\,)} \tag{3.70}$$

with

$$w^{\mu\nu}_{(\,)} = \int \frac{d^3k}{2E_k(2\pi)^3} h^{\mu\nu}_{(\,)} (2\pi)^4 \delta^4(p_A + p_B - q - k) \tag{3.71}$$

Here k is the momentum of the unobserved particle: a gluon in the case of the real emission subprocess and a quark for the Compton subprocess. $h^{\mu\nu}_{(\,)}$ is defined as the appropriate partonic tensor in four dimensions. The empty parenthesis in equations (3.70) and (3.71) can be filled with the appropriate partonic labels: $j\bar{j}$, gj or jg. The phase space can be rewritten

$$\begin{aligned}\int \frac{d^3k}{2E_k(2\pi)^3}(2\pi)^4\delta^4(p_A+p_B-q-k) &= 2\pi\int d^4k\delta^4(p_A+p_B-q-k)\delta(k^2)\\ &= 2\pi\delta\left[(p_A+p_B-q)^2\right]\end{aligned}$$

and with the help of the parton Mandelstam variables

$$s = (p_B+p_A)^2 \tag{3.72}$$

$$t = (p_B-q)^2 \tag{3.73}$$

$$u = (p_A-q)^2 \tag{3.74}$$

we get

$$2\pi\delta\left(s+t+u-Q^2\right) \tag{3.75}$$

Now, we can express the momentum q^μ in terms of the rapidity y, Eq.(3.10), the invariant mass Q^2 and the transverse momentum Q_T^2;

$$q^\mu = \left(\sqrt{Q^2+Q_T^2}\cosh y, Q_T, 0, \sqrt{Q^2+Q_T^2}\sinh y\right) \tag{3.76}$$

which allows us to rewrite s, t and u:

$$s = \frac{Q^2}{z_A z_B} \tag{3.77}$$

$$t = Q^2 - Q^2\frac{1}{z_B}\sqrt{1+\frac{Q_T^2}{Q^2}} \tag{3.78}$$

$$u = Q^2 - Q^2\frac{1}{z_A}\sqrt{1+\frac{Q_T^2}{Q^2}} \tag{3.79}$$

with

$$\begin{aligned} z_A &= \frac{x_A}{\xi_A}\\ z_B &= \frac{x_B}{\xi_B}\end{aligned} \tag{3.80}$$

where, x_A and x_B were defined in Eq.(3.13).

The delta function (3.75) can also be reparametrized

$$\frac{2\pi}{S}\delta\left[\left(\xi_A - x_A\sqrt{1+\frac{Q_T^2}{Q^2}}\right)\left(\xi_B - x_B\sqrt{1+\frac{Q_T^2}{Q^2}}\right) - x_A x_B\frac{Q_T^2}{Q^2}\right] \tag{3.81}$$

and here we have also used

$$s = \xi_A \xi_B S \tag{3.82}$$

$$x_A x_B = \frac{Q^2}{S} \tag{3.83}$$

For the real emission subprocess $q\bar{q}$ the parton tensor $h_{j\bar{j}}^{\mu\nu(R)}$ is, Eq.(B.7), [87], [108]:

$$\begin{aligned} h_{j\bar{j}}^{\mu\nu(R)} = \tfrac{4}{9} & \frac{e_j^2 g^2}{ut} \left\{ -4Q^2 \left(p_A^\mu p_A^\nu + p_B^\mu p_B^\nu \right) - \left[\left(Q^2 - t \right)^2 + \left(Q^2 - u \right)^2 \right] g^{\mu\nu} \right. \\ + & \left. 2 \left(Q^2 - t \right) \left(p_B^\mu q^\nu + p_B^\nu q^\mu \right) + 2 \left(Q^2 - u \right) \left(p_A^\mu q^\nu + p_A^\nu q^\mu \right) \right\} \end{aligned} \tag{3.84}$$

with 4/9 is the "color factor", e_q is the electric charge of the participating quark (or antiquark) in units of e and g is the strong scale factor. For the Compton subprocess qg we find in Eq.(B.10), [87], [108]:

$$\begin{aligned} h_{jg}^{\mu\nu} = \frac{1}{6} \frac{e_j^2 g^2}{us} \{ & \quad 8Q^2 p_A^\mu p_A^\nu + 4Q^2 p_B^\mu p_B^\nu + 4Q^2 \left(p_A^\mu p_B^\nu + p_B^\mu p_A^\nu \right) \\ + & \left[\left(Q^2 - s \right)^2 + \left(Q^2 - u \right)^2 \right] g^{\mu\nu} - 2 \left(Q^2 + t + 2s \right) \left(p_A^\mu q^\nu + q^\mu p_A^\nu \right) \\ - & \ 2 \left(Q^2 + s \right) \left(p_B^\mu q^\nu + q^\mu p_B^\nu \right) + 4s q^\mu q^\nu \ \} \end{aligned} \tag{3.85}$$

and for the exchanged process gq Eq.(B.11):

$$\begin{aligned} h_{gj}^{\mu\nu} = \frac{1}{6} \frac{e_j^2 g^2}{ts} \{ & \quad 8Q^2 p_B^\mu p_B^\nu + 4Q^2 p_A^\mu p_A^\nu + 4Q^2 \left(p_A^\mu p_B^\nu + p_B^\mu p_A^\nu \right) \\ + & \left[\left(Q^2 - s \right)^2 + \left(Q^2 - t \right)^2 \right] g^{\mu\nu} - 2 \left(Q^2 + u + 2s \right) \left(p_B^\mu q^\nu + q^\mu p_B^\nu \right) \\ - & \ 2 \left(Q^2 + s \right) \left(p_A^\mu q^\nu + q^\mu p_A^\nu \right) + 4s q^\mu q^\nu \ \} \end{aligned} \tag{3.86}$$

Putting all the previous results together we can conclude with the prediction of pQCD to next-to-leading-order for the hadronic tensor

$$\begin{aligned} W_{()}^{\mu\nu} = & \ 2\pi \sum_j \int_{x_A}^1 \frac{d\xi_A}{\xi_A} \int_{x_B}^1 \frac{d\xi_B}{\xi_B} \delta \left[\left(\xi_A - x_A \sqrt{1 + \frac{Q_T^2}{Q^2}} \right) \left(\xi_B - x_B \sqrt{1 + \frac{Q_T^2}{Q^2}} \right) - x_A x_B \frac{Q_T^2}{Q^2} \right] \\ & \times f_{()/A} \left(\xi_A ; \mu \right) f_{()/B} \left(\xi_B ; \mu \right) h_{()}^{\mu\nu} \end{aligned} \tag{3.87}$$

with the sum over all quark and antiquark flavors.

3.4 QCD predictions for Drell-Yan

After a long detour we are finally able to give the NLO predictions of pQCD for the structure functions. It is worth to notice that these functions are frame dependent. This is built in the definitions for the projection operators given in Eq.(2.21), since the vectors $\hat{x}$ and $\hat{z}$ are particular of the basis chosen in the lepton c.m.s. As it was said before, we will be working in the Collins-Soper frame, see Section 2.1. In this system we have for $\hat{z}$ and $\hat{x}$ in terms of quantities measured in the hadron c.m.s.:

$$\hat{z}^\mu = \begin{pmatrix} \sinh y \\ 0 \\ 0 \\ \cosh y \end{pmatrix} \quad (3.88) \qquad \hat{x}^\mu = \begin{pmatrix} \frac{Q_T}{Q}\cosh y \\ \sqrt{1+\frac{Q_T^2}{Q^2}} \\ 0 \\ \frac{Q_T}{Q}\sinh y \end{pmatrix} \quad (3.89) \qquad \hat{y}^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (3.90)$$

With these vectors we can evaluate the necessary inner products

$$\begin{aligned} \hat{z}\cdot p_A &= -\frac{\sqrt{S}}{2}\xi_A\, e^{-y} \\ &= -\frac{Q}{2z_A} \qquad (3.91) \end{aligned} \qquad \begin{aligned} \hat{z}\cdot p_B &= \frac{\sqrt{S}}{2}\xi_B\, e^{y} \\ &= \frac{Q}{2z_B} \qquad (3.93) \end{aligned}$$

$$\begin{aligned} \hat{x}\cdot p_A &= \frac{\sqrt{S}}{2}\frac{Q_T}{Q}\xi_A\, e^{-y} \\ &= \frac{Q_T}{2z_A} \qquad (3.92) \end{aligned} \qquad \begin{aligned} \hat{x}\cdot p_B &= \frac{\sqrt{S}}{2}\frac{Q_T}{Q}\xi_B\, e^{y} \\ &= \frac{Q_T}{2z_B} \qquad (3.94) \end{aligned}$$

where we have used equations (3.6), (3.13), (3.42), (3.43) and (3.80). Remember also that by construction (regardless of the chosen dilepton c.m.s)

$$\begin{aligned} \hat{z}\cdot q &= 0 \\ \hat{x}\cdot q &= 0 \\ \hat{y}\cdot q &= 0 \end{aligned} \qquad (3.95)$$

We are now ready to evaluate the projection operators. First, we can start with the real emission contributions. In order to make the notation manageable we will omit from the real structure functions the following overall factor[17]:

$$\frac{\alpha_S(\mu)}{\pi} \quad \frac{(2\pi)^4}{3} \quad \sum_j e_j^2 \int_{x_A}^1 \frac{d\xi_A}{\xi_A} \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{j/A}(\xi_A;\mu) f_{\bar{j}/B}(\xi_B;\mu)$$
$$\times\delta\left[\left(\xi_A - x_A\sqrt{1+\frac{Q_T^2}{Q^2}}\right)\left(\xi_B - x_B\sqrt{1+\frac{Q_T^2}{Q^2}}\right) - x_A x_B \frac{Q_T^2}{Q^2}\right] \quad (3.96)$$

thus,

$$\boxed{\begin{array}{lcl} w_L^R & = & \frac{2}{3\pi}\left(\frac{z_A}{z_B}+\frac{z_B}{z_A}\right) \\ w_T^R & = & \frac{1}{3\pi}\left(1+\frac{2Q^2}{Q_T^2}\right)\left(\frac{z_A}{z_B}+\frac{z_B}{z_A}\right) \\ w_{\Delta\Delta}^R & = & \frac{1}{2}W_L^R \\ w_\Delta^R & = & \frac{2}{3\pi}\frac{Q}{Q_T}\left(\frac{z_A}{z_B}-\frac{z_B}{z_A}\right) \end{array}}$$

Table 3.3 NLO predictions for structure functions, real contribution

where we have exploited the relation

$$ut = sQ_T^2 \quad (3.97)$$

and Eq.(2.21).

Table 3.3 is equivalent to write for $\frac{dN}{d\Omega}$, see Eq.(2.23) and [25], [40] :

$$\frac{dN}{d\Omega} = \frac{3}{16\pi}\left[\frac{Q^2+\frac{3}{2}Q_T^2}{Q^2+Q_T^2} + \frac{Q^2-\frac{1}{2}Q_T^2}{Q^2+Q_T^2}\cos^2\theta + \frac{1}{2}\frac{Q_T^2}{Q^2+Q_T^2}\cos 2\phi\sin^2\theta\right.$$
$$\left. + \frac{QQ_T}{Q^2+Q_T^2}K\left(x_A, x_B, \frac{Q_T}{S}\right)\sin 2\theta\cos\phi\right] \quad (3.98)$$

with

$$K\left(x_A, x_B, \frac{Q_T}{S}\right) = \frac{\frac{z_A}{z_B}-\frac{z_B}{z_A}}{\frac{z_A}{z_B}+\frac{z_B}{z_A}} \quad (3.99)$$

where the factor Eq.(3.96) has been omitted in the numerator and denominator. Equation (3.98) can be used to find λ, μ and ν [25], and they are summarized in Table 3.2.

[17]Here we have used $\frac{g^2}{4\pi} = \alpha_S$

λ	$=$	$\frac{Q^2-\frac{1}{2}Q_T^2}{Q^2+\frac{3}{2}Q_T^2}$
μ	$=$	$\frac{QQ_T}{Q^2+\frac{3}{2}Q_T^2}K\left(x_A,x_B,\frac{Q_T}{S}\right)$
ν	$=$	$\frac{Q_T^2}{Q^2+\frac{3}{2}Q_T^2}$

Table 3.4 NLO predictions for λ, μ and ν, real contribution

Notice that since the terms in denominator and numerator of $K\left(x_A,x_B,\frac{Q_T}{S}\right)$ are different and each one of them are in convolution with the parton distribution functions it is not possible to cancel them as it was done for λ and ν. Interestingly these two parameters are independent of the parton densities. Probably more interesting is the fact that the Lam-Tung relation still holds:

$$1-\lambda-2\nu=0 \tag{3.100}$$

which can already been seen in the Table 3.3, since $W^R_{\Delta\Delta}=\frac{1}{2}W^R_L$.

Similarly, we can now write the overall factor for the helicity structure functions for the Compton subprocess qg

$$\frac{\alpha_S(\mu)}{\pi}\frac{(2\pi)^4}{3}\left(-\frac{Q^2}{Q_T^2}\right)\sum_j e_j^2\int_{x_A}^1\frac{d\xi_A}{\xi_A}\int_{x_B}^1\frac{d\xi_B}{\xi_B}\left[\frac{\xi_B\sqrt{1+\frac{Q_T^2}{Q^2}}-x_B}{x_B}\right]f_{j/A}(\xi_A;\mu)\,f_{g/B}(\xi_B;\mu)$$
$$\times\delta\left[\left(\xi_A-x_A\sqrt{1+\frac{Q_T^2}{Q^2}}\right)\left(\xi_B-x_B\sqrt{1+\frac{Q_T^2}{Q^2}}\right)-x_Ax_B\frac{Q_T^2}{Q^2}\right] \tag{3.101}$$

where we have employed

$$u=-Q_T^2\frac{\xi_A}{x_A}\left(\frac{\xi_B}{\xi_B\sqrt{1+\frac{Q_T^2}{Q^2}}-x_B}\right) \tag{3.102}$$

The helicity structure functions are:

The DY predictions for the same process in terms of λ, μ and ν can be found in Table 3.6

where the overall factor, Eq.(3.101), has been omitted in the numerator and denominator. Similarly to the factor $K\left(x_A,x_B,\frac{Q_T}{S}\right)$in Table (3.98) we cannot cancel any terms between

w_L^C	$=$	$\frac{1}{4\pi}\left\{(z_A-z_B)^2+z_B^2-\left[(1-z_Az_B)^2+z_B^2\right]-z_B^2\frac{Q_T^2}{Q^2}\right\}$
w_T^C	$=$	$-\frac{1}{8\pi}\left[(z_A-z_B)^2+z_B^2+(1-z_Az_B)^2+z_B^2+z_B^2\frac{Q_T^2}{Q^2}\right]$
$w_{\Delta\Delta}^C$	$=$	$\frac{1}{2}W_L^C$
w_{Δ}^C	$=$	$-\frac{1}{4\pi}\frac{Q_T}{Q}\left(z_A^2-z_B^2\right)$

Table 3.5 NLO predictions for structure functions, Compton contribution qg

λ	$=$	$\frac{3\left[(z_A-z_B)^2+z_B^2\right]-\left[(1-z_Az_B)^2+z_B^2\right]-z_B^2\frac{Q_T^2}{Q^2}}{3\left[(1-z_Az_B)^2+z_B^2\right]-\left[(z_A-z_B)^2+z_B^2\right]+3z_B^2\frac{Q_T^2}{Q^2}}$
μ	$=$	$\frac{2\frac{Q_T}{Q}(z_A^2-z_B^2)}{3\left[(1-z_Az_B)^2+z_B^2\right]-\left[(z_A-z_B)^2+z_B^2\right]+3z_B^2\frac{Q_T^2}{Q^2}}$
ν	$=$	$\frac{-2\left\{(z_A-z_B)^2+z_B^2-\left[(1-z_Az_B)^2+z_B^2\right]-z_B^2\frac{Q_T^2}{Q^2}\right\}}{3\left[(1-z_Az_B)^2+z_B^2\right]-\left[(z_A-z_B)^2+z_B^2\right]+3z_B^2\frac{Q_T^2}{Q^2}}$

Table 3.6 NLO predictions for λ, μ and ν, Compton contribution qg

numerator and denominator in order to simplify the structure functions.

To finish, we include the overall factor of the Compton process gq

$$\frac{\alpha_S(\mu)}{\pi}\frac{(2\pi)^4}{3}\left(-\frac{Q^2}{Q_T^2}\right)\sum_j e_j^2\int_{x_A}^1\frac{d\xi_A}{\xi_A}\int_{x_B}^1\frac{d\xi_B}{\xi_B}\left[\frac{\xi_A\sqrt{1+\frac{Q_T^2}{Q^2}}-x_A}{x_A}\right]f_{g/A}(\xi_A;\mu)\,f_{j/B}(\xi_B;\mu)$$
$$\times\delta\left[\left(\xi_A-x_A\sqrt{1+\frac{Q_T^2}{Q^2}}\right)\left(\xi_B-x_B\sqrt{1+\frac{Q_T^2}{Q^2}}\right)-x_Ax_B\frac{Q_T^2}{Q^2}\right] \tag{3.103}$$

with

$$t=-Q_T^2\frac{\xi_B}{x_B}\left(\frac{\xi_A}{\xi_A\sqrt{1+\frac{Q_T^2}{Q^2}}-x_A}\right) \tag{3.104}$$

The respective helicity structure functions:

$$
\begin{array}{lcl}
w_L^C & = & \frac{1}{4\pi}\left\{(z_B - z_A)^2 + z_A^2 - \left[(1 - z_A z_B)^2 + z_A^2\right] - z_A^2 \frac{Q_T^2}{Q^2}\right\} \\
w_T^C & = & -\frac{1}{8\pi}\left[(z_B - z_A)^2 + z_A^2 + (1 - z_A z_B)^2 + z_A^2 + z_A^2 \frac{Q_T^2}{Q^2}\right] \\
w_{\Delta\Delta}^C & = & \frac{1}{2} W_L^C \\
w_\Delta^R & = & -\frac{1}{4\pi}\frac{Q_T}{Q}\left(z_B^2 - z_A^2\right)
\end{array}
$$

Table 3.7 NLO predictions for structure functions, Compton contribution gq

and values for λ, μ and ν can be seen in Table 3.8. Where once again the overall factor,

$$
\begin{array}{lcl}
\lambda & = & \dfrac{3\left[(z_B - z_A)^2 + z_A^2\right] - \left[(1 - z_A z_B)^2 + z_A^2\right] - z_A^2 \frac{Q_T^2}{Q^2}}{3\left[(1 - z_A z_B)^2 + z_B^2\right] - \left[(z_B - z_A)^2 + z_A^2\right] + 3 z_A^2 \frac{Q_T^2}{Q^2}} \\
\mu & = & \dfrac{2\frac{Q_T}{Q}(z_B^2 - z_A^2)}{3\left[(1 - z_A z_B)^2 + z_A^2\right] - \left[(z_B - z_A)^2 + z_A^2\right] + 3 z_A^2 \frac{Q_T^2}{Q^2}} \\
\nu & = & \dfrac{-2\left\{(z_B - z_A)^2 + z_A^2 - \left[(1 - z_A z_B)^2 + z_A^2\right] - z_A^2 \frac{Q_T^2}{Q^2}\right\}}{3\left[(1 - z_A z_B)^2 + z_A^2\right] - \left[(z_B - z_A)^2 + z_A^2\right] + 3 z_A^2 \frac{Q_T^2}{Q^2}}
\end{array}
$$

Table 3.8 NLO predictions for λ, μ and ν, Compton contribution qg

Eq.(3.103), has been omitted in the numerator and denominator since no further simplification is possible.

In Fig. 3.5, 3.9 and 3.13 we show the NLO predictions for the parameters λ, μ and ν with $Q = 10\ \frac{GeV}{c}$, $\sqrt{S} = 800\ \frac{GeV}{c}$ and $y = 0$ in the Collins-Soper frame:

A few comments are now relevant. Observing figures 3.5 through 3.8 we first notice that many of the central values for λ in the data sets from E615 and E866 are above 1 which contradicts the simple fact that the physical range for this parameter is $-1 \leq \lambda \leq 1$, see Eq.(2.26). We can also remark that there is a qualitative agreement between the set from NA10 and the perturbative predictions. A good match is not expected since the Q dependence has not been

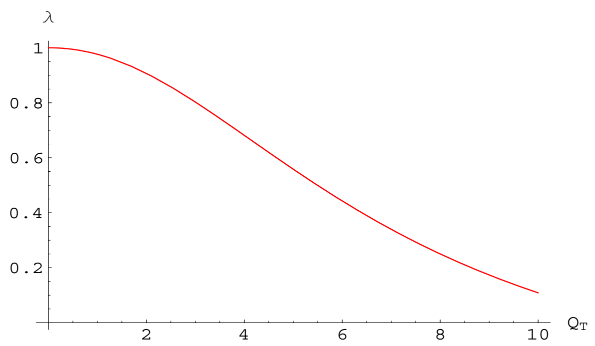


Figure 3.5 λ vs Q_T NLO prediction

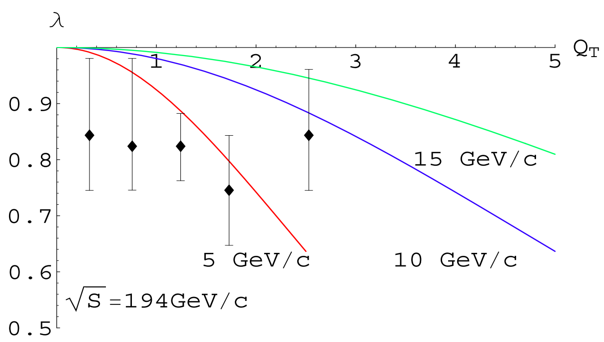


Figure 3.6 λ vs Q_T for NLO and NA10

taken into account.

For the ν parameter we note that the next-to-leading-order prediction is consistently below the experimental data of NA10 and E615, see figures 3.9,3.10 and 3.11. For the E866 we have that the theoretical curves go through the experimental points, Fig. 3.12. Comparing Fig. 3.13 with Fig. 1.2 we see that the prediction $\mu \approx 0$ is compatible with the experimental results from all three collaborations. We believe that at least part of the poor correspondence between experiments and NLO predictions can be attributed to the fact that the experimental results are integrated over Q which in the case of E866 includes a range between $4.5 < Q < 9$ GeV/c and $Q > 10.7$ GeV/c [128].

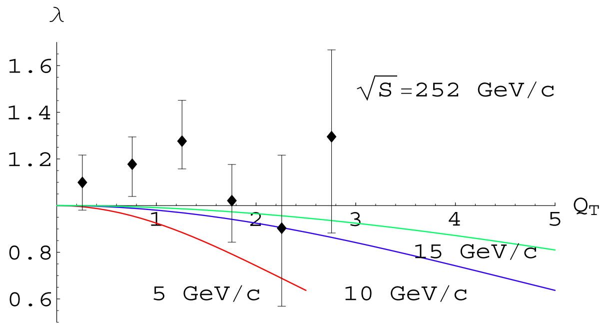


Figure 3.7 λ vs Q_T for NLO and E615

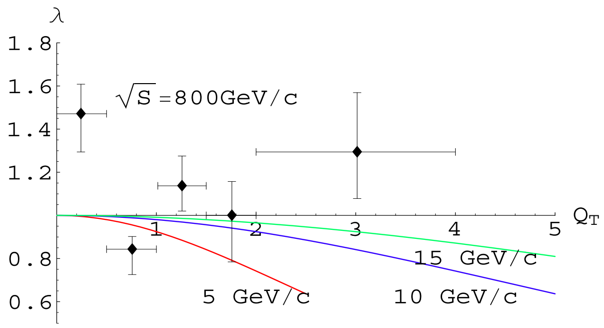


Figure 3.8 λ vs Q_T for NLO and E866

We notice here one prediction more coming from the NLO calculations. The Lam-Tung relation holds also for the Compton subprocess since $W^C_{\Delta\Delta}=\frac{1}{2}W^C_L$ in both Table 3.6 and Table 3.8. This means that the Lam-Tung relation is valid for the NLO corrections for the whole range[18] of allowed Q_T [25] and [88]. It is also worthy of attention that despite the fact that we have only worked in the CS frame, the Lam-Tung relation is valid at NLO independently of the dilepton c.m.s. chosen [88]. To see this, observe first:

$$h^{\mu\nu(R)}_{q\bar{q}}(-g_{\mu\nu}) = \frac{4e_j^2g^2}{9}\frac{2}{ut}\left[(Q^2-t)^2+(Q^2-u)^2\right] \tag{3.105}$$

$$h^{\mu\nu}_{qg}(-g_{\mu\nu}) = \frac{e_j^2g^2}{6}\frac{2}{s(-u)}\left[(Q^2-s)^2+(Q^2-u)^2\right] \tag{3.106}$$

$$h^{\mu\nu}_{gq}(-g_{\mu\nu}) = \frac{e_j^2g^2}{6}\frac{2}{s(-t)}\left[(Q^2-s)^2+(Q^2-t)^2\right] \tag{3.107}$$

[18]The reader should remember that we have assumed that $Q_T >> \Lambda_{\rm QCD}$.

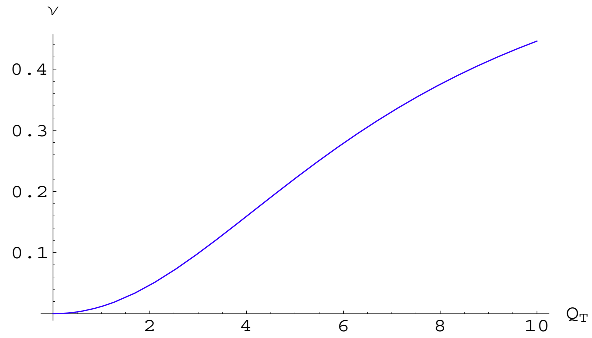


Figure 3.9 ν vs Q_T NLO prediction

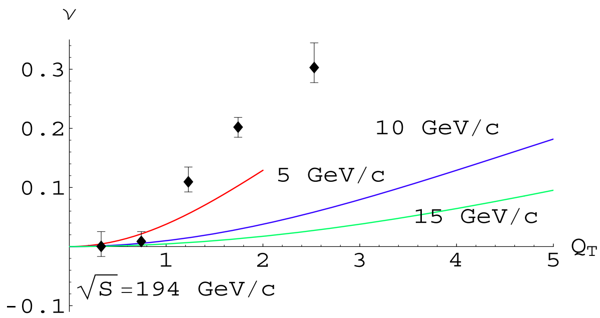


Figure 3.10 ν vs Q_T for NLO and NA10

and then compare with the coefficients of $-g^{\mu\nu}$ in Eqs.(3.84), (3.86) and (3.86). Thus, we deduce from equations (3.22) - (3.28):

$$W^{\mu\nu(R)}_{q\bar{q}}(-g_{\mu\nu}) = 2(G_1)_{q\bar{q}} \tag{3.108}$$

$$W^{\mu\nu}_{qg}(-g_{\mu\nu}) = 2(G_1)_{qg} \tag{3.109}$$

$$W^{\mu\nu}_{gq}(-g_{\mu\nu}) = 2(G_1)_{gq} \tag{3.110}$$

and by the discussion that follows Eq.(3.22) we can conclude that[19]

$$1 - \lambda - 2\nu = 0 \tag{3.111}$$

regardless of the particular dilepton frame used. It is important to remember here that E866 and NA10 are largely compatible with the Lam-Tung relation as can be appreciated in Fig. 1.2.

[19]This also can be seen easily since G_1 is an invariant structure function

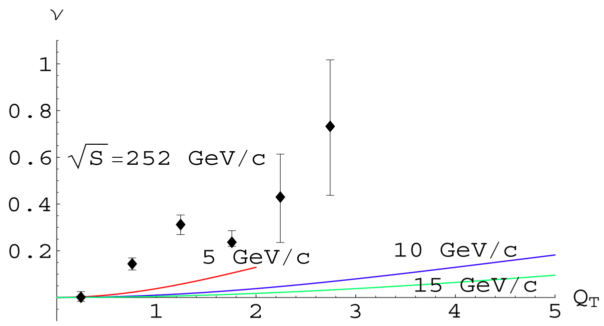


Figure 3.11 ν vs Q_T for NLO and E615

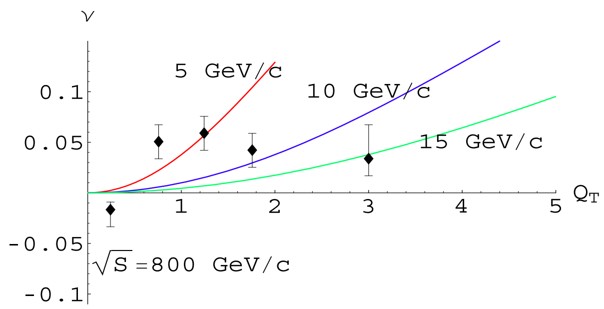


Figure 3.12 ν vs Q_T for NLO and E886

We finish this chapter with figures 3.14, 3.15 and 3.16 for the NLO predictions for the $W_T, W_L, W_{\Delta\Delta}$ structure functions in a $p\bar{p}$ collision with $Q = 10\ \frac{GeV}{c}$, $\sqrt{S} = 800\ \frac{GeV}{c}$ and $y = 0$. The green line denotes $W_i = W_{i\ q\bar{q}} + W_{i\ qg} + W_{i\ gq}$, the red one is for $W_i = W_{i\ q\bar{q}}$ and blue for $W_{i\ qg}$ with $i = T,\ L,\ \Delta\Delta$

For the last two pictures, figures 3.17 and 3.17, we have the same parameters in the phase space used for the previous plots but we have changed the colors to blue for $W_{qg\ \Delta}$ and red for $W_\Delta = W_{\Delta\ q\bar{q}} + W_{\Delta\ qg} + W_{\Delta\ gq}$. There are a couple of important remarks here. As it can be seen in Fig. 3.18 the contribution from the $q\bar{q}$-process is not positive definite and it oscillates in sign wildly as a function of Q_T, this is the only structure function with such behavior. Observing the total $W_{\Delta\ q\bar{q}}$ (red line), we can conclude that $W_{\Delta\ q\bar{q}}$ decreases the total value of the single delta structure function that is still positive definite as it is expected from the definition.

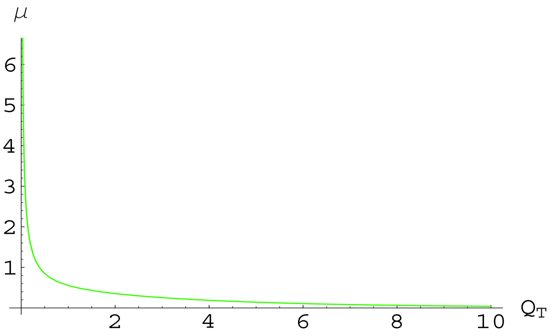


Figure 3.13 μ vs Q_T NLO prediction

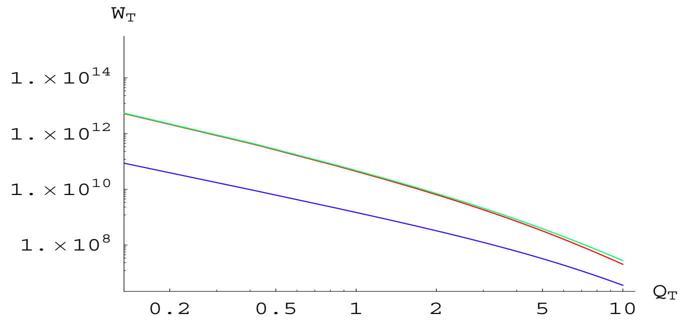


Figure 3.14 W_T vs Q_T, NLO prediction

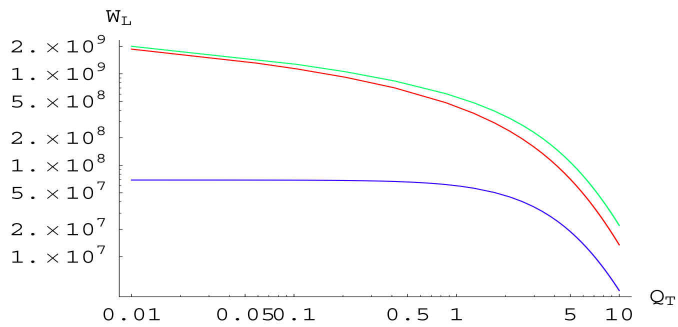


Figure 3.15 W_L vs Q_T, NLO prediction

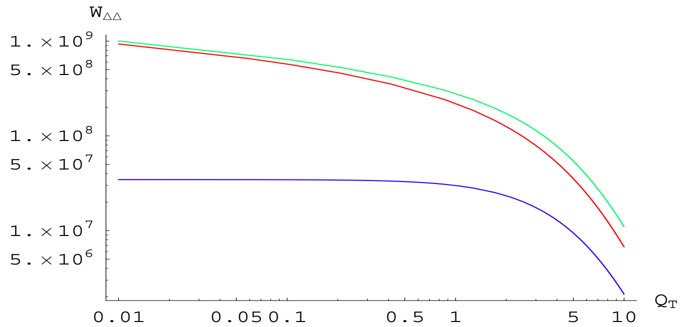


Figure 3.16 $W_{\Delta\Delta}$ vs Q_T, NLO prediction

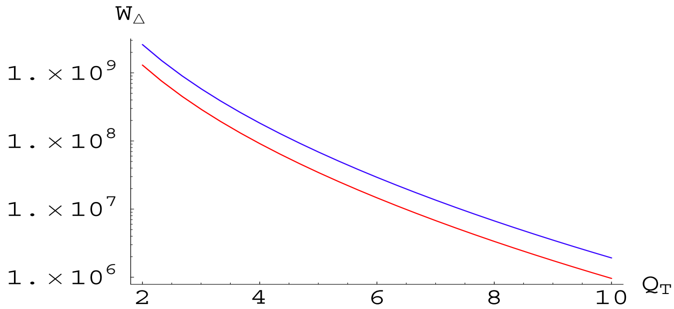


Figure 3.17 W_{Δ} vs Q_T, NLO prediction

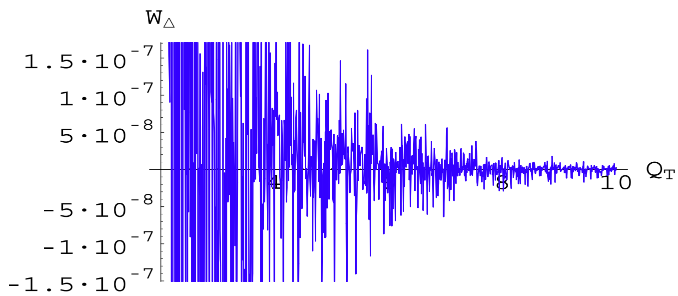


Figure 3.18 $W_{\Delta\, q\bar{q}}$ vs Q_T, NLO prediction

CHAPTER 4. LOW Q_T LIMIT AND RESUMMATION

Resummation is the organization of soft and collinear radiation to all orders in perturbation theory [120]. In this chapter we will apply this technique to the inclusive cross section of the DY process and later in the next chapter we will extend it to the fully differential cross section.

4.1 Low Q_T limit

In the last chapter the relevant results were calculated under the assumption that $\Lambda_{QCD} \ll Q \approx Q_T$. But most of the experimental data lies in the region $0 \leq Q_T < Q$. This fact forces us to evaluate the limit of low Q_T for the structure functions and for λ, μ, ν.

Let us begin with the structure functions for the real contribution. Using the results from Table 3.3 and Eq.(3.96) we can find, for example, in the case of W_T^R,

$$\begin{aligned}\lim_{Q_T \to 0} W_T^R &= \lim_{Q_T \to 0} \frac{32\pi^3}{9} \left(\frac{\alpha_S(\mu)}{\pi} \right) \frac{Q^2}{Q_T^2} \sum_j e_j^2 \int_{x_A}^1 \frac{d\xi_A}{\xi_A} \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{j/A}(\xi_A;\mu) f_{\bar{j}/B}(\xi_B;\mu) \\ &\times \left(\frac{z_A}{z_B} + \frac{z_B}{z_A} \right) \delta \left[(\xi_A - x_A)(\xi_B - x_B) - x_A x_B \frac{Q_T^2}{Q^2} \right] \end{aligned} \tag{4.1}$$

Observing the delta function in the above expression we can easily identify the regions where the $1/Q_T^2$ divergence comes from, i.e regions[1] where $Q_T \to 0$:

1. Region where $\xi_A - x_A \to 0$ or $z_A \to 1$ with $\xi_B - x_B$ different from zero and constant;

2. Region where $\xi_B - x_B \to 0$ or $z_B \to 1$ with $\xi_A - x_A$ different from zero and constant;

3. Region where $(\xi_B - x_B), (\xi_A - x_A) \to 0$ or $z_A, z_B \to 1$;

[1] These regions define the regions of integration for the expansion of the delta function, see Fig.C.1

Using Eq.(C.4) we can find the asymptotic expansion for the integral,

$$\begin{aligned}
&\lim_{Q_T\to 0} W_T^R \\
&= \lim_{Q_T\to 0}\left(\frac{\alpha_S(Q)}{\pi}\right)\frac{64\pi^4}{9}\frac{1}{2\pi}\frac{S}{Q_T^2}\sum_j e_j^2 \\
&\quad\times\left\{f_{\bar{j}/B}(x_B)\int_{x_A}^1\frac{d\xi_A}{\xi_A}f_{j/A}(\xi_A)\frac{1}{\xi_A}\frac{\xi_A^2+x_A^2}{(\xi_A-x_A)_+}\right. \\
&\quad +2f_{j/A}(x_A)f_{\bar{j}/B}(x_B)\ln\left[\frac{(1-x_A)(1-x_B)S}{Q_T^2}\right] \\
&\quad\left.+f_{j/A}(x_A)\int_{x_B}^1\frac{d\xi_B}{\xi_B}f_{\bar{j}/B}(\xi_B)\frac{1}{\xi_B}\frac{\xi_B^2+x_B^2}{(\xi_B-x_B)_+}\right\}
\end{aligned} \tag{4.2}$$

which can be rewritten (after omitting the parton flux factor) as:

$$\begin{aligned}
&\lim_{Q_T\to 0} W_T^R \\
&= \lim_{Q_T\to 0}\frac{(2\pi)^4}{3}S\sum_j e_j^2\left(\frac{\alpha_S(Q)}{\pi}\right)\frac{1}{2\pi}\frac{1}{Q_T^2} \\
&\quad\times\left\{\delta(1-z_B)\frac{4}{3}\left[\frac{1+z_A^2}{1-z_A}\right]_+ +\delta(1-z_A)\frac{4}{3}\left[\frac{1+z_B^2}{1-z_B}\right]_+\right. \\
&\quad\left.+\,2\delta(1-z_A)\delta(1-z_B)\left[\frac{4}{3}\ln\left(\frac{Q^2}{Q_T^2}\right)-2\right]\right\}
\end{aligned} \tag{4.3}$$

where Eq.(C.8) was used. See figures 4.1 and 4.2 for the properties of the asymptotic behavior of the NLO prediction in $p\bar{p}$ with $Q = 10\ \frac{GeV}{c}$, $\sqrt{S} = 800\ \frac{GeV}{c}$ and $y = 0$:

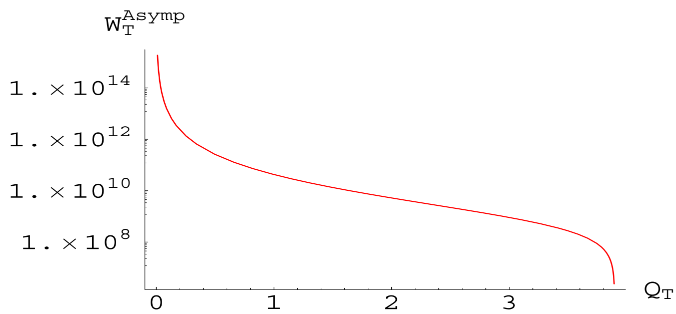


Figure 4.1 $W^{Asymp}_{T\,q\bar{q}}$ vs Q_T, $0.1 \le Q_T \le 4$ NLO prediction

Notice that in Eq.(4.3) the regions where the divergences are present have been made explicit and a divergence proportional only to $\ln\left(\frac{Q^2}{Q_T^2}\right)$ has been left neglected. We observed from

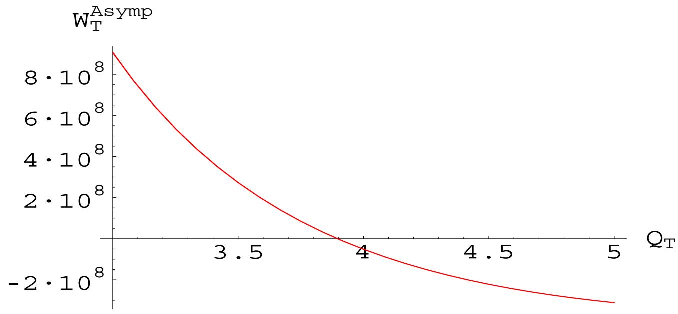


Figure 4.2 $W_{T\,q\bar{q}}^{Asymp}$ vs Q_T, $3 \leq Q_T \leq 5$ NLO prediction

Table 3.3 that W_L^R and $W_{\Delta\Delta}^R$ have a similar singularity, which is integrable. Their contribution at low Q_T is negligible compared with the singularities of W_T^R. Therefore, the cross section in the limit $Q_T \to 0$ is dominated by the transverse structure function.

W_Δ^R has a singularity proportional to $1/Q_T$ as can be seen below:

$$\begin{aligned}
&\lim_{Q_T\to 0} W_\Delta^R \\
&= \lim_{Q_T\to 0}\left(\frac{\alpha_S(Q)}{\pi}\right)\frac{64\pi^4}{9}\frac{1}{2\pi}\frac{S}{Q_TQ}\sum_j e_j^2 \\
&\quad\times\left\{f_{j/A}(x_A)\int_{x_B}^1 d\xi_B\, f_{\bar{j}/B}(\xi_B)\frac{\xi_B+x_B}{\xi_B^2} - f_{\bar{j}/B}(x_B)\int_{x_A}^1 d\xi_A\, f_{j/A}(\xi_A)\frac{\xi_A+x_A}{\xi_A^2}\right\} \quad (4.4)\\
&= \lim_{Q_T\to 0}\frac{(2\pi)^4}{3}S\sum_j e_j^2\left(\frac{\alpha_S(Q)}{\pi}\right)\frac{1}{2\pi}\frac{1}{Q_TQ}\left\{\delta(1-z_A)\frac{4}{3}(1+z_B)\right. \\
&\quad\left.-\delta(1-z_B)\frac{4}{3}(1+z_A)\right\} \quad (4.5)
\end{aligned}$$

Note the absence of the logarithmic singularity which is present in all the other structure functions.

The limits for λ, μ, ν real contribution are easily obtained from Table 3.4:

$$\begin{aligned}
\lim_{Q_T\to 0}\lambda &= 1 \\
\lim_{Q_T\to 0}\mu &= 0 \\
\lim_{Q_T\to 0}\nu &= 0 \quad (4.6)
\end{aligned}$$

which just recovers the parton model prediction.

For the Compton subprocess contributions, Tables 3.5 and 3.8, we first observe that $\lim_{Q_T\to 0} W_L^C$ is zero or finite for any of the regions where $Q_T \to 0$, this of course is also valid for $W_{\Delta\Delta}^C$. For the qg subprocess, Table 3.5, the only region that contains a divergence is $z_A \to 1$ and we have as limits:

$$\lim_{Q_T\to 0} W_T^C = \lim_{Q_T\to 0} \frac{(2\pi)^4}{3} S \sum_j e_j^2 \left(\frac{\alpha_S(Q)}{\pi}\right) \frac{1}{2\pi} \frac{1}{Q_T^2} \left\{\frac{1}{2}\left[(1-z_B)^2 + z_B^2\right] \delta(1-z_A)\right\} \tag{4.7}$$

and

$$\lim_{Q_T\to 0} W_\Delta^C = \lim_{Q_T\to 0} \frac{(2\pi)^4}{3} S \sum_j e_j^2 \left(\frac{\alpha_S(Q)}{\pi}\right) \frac{1}{2\pi} \frac{1}{Q_T Q} \left\{\frac{1}{2}\left(1-z_B^2\right) \delta(1-z_A)\right\} \tag{4.8}$$

likewise we can obtain, from Table 3.5, for the gq process, in the $z_B \to 1$ region:

$$\lim_{Q_T\to 0} W_T^C = \lim_{Q_T\to 0} \frac{(2\pi)^4}{3} S \sum_j e_j^2 \left(\frac{\alpha_S(Q)}{\pi}\right) \frac{1}{2\pi} \frac{1}{Q_T^2} \left\{\frac{1}{2}\left[(1-z_A)^2 + z_A^2\right] \delta(1-z_B)\right\} \tag{4.9}$$

and

$$\lim_{Q_T\to 0} W_\Delta^C = \lim_{Q_T\to 0} \frac{(2\pi)^4}{3} S \sum_j e_j^2 \left(\frac{\alpha_S(\mu)}{\pi}\right) \frac{1}{2\pi} \frac{1}{Q_T Q} \left\{\frac{1}{2}\left(1-z_A^2\right) \delta(1-z_B)\right\} \tag{4.10}$$

The existence of the singularities proportional to $1/Q_T^2$ and $\ln\left(\frac{Q^2}{Q_T^2}\right)/Q_T^2$ in the low Q_T^2 limit spoils the usefulness of the perturbative expansion in this region. We will see in Sec. (4.3.1) how resummation handles this problem to all orders, but first we turn our attention to the physical origin of the singularities just observed. We finish this section with a graphic example of the behavior of $W_T\ qg$ for $p\bar{p}$ with $Q = 10\ \frac{GeV}{c}$, $\sqrt{S} = 800\ \frac{GeV}{c}$ and $y = 0$:

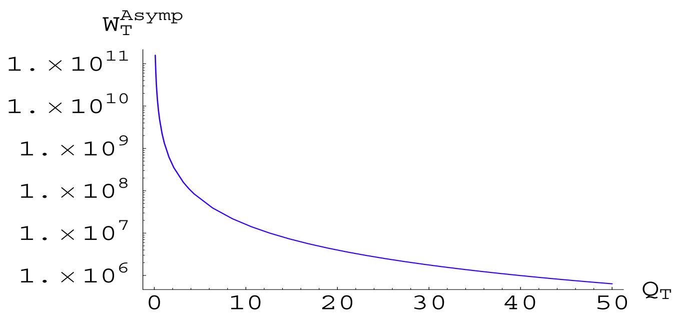


Figure 4.3 $W_{T\,qg}^{Asymp}$ vs Q_T, NLO prediction

4.2 Origin of the singularities

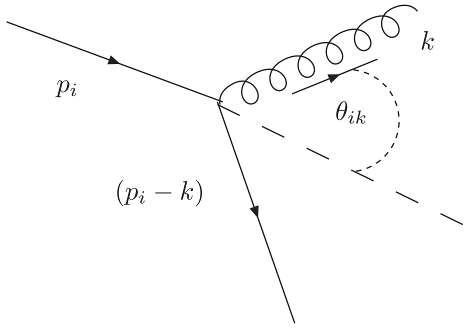
The divergences that we have observed above are typical examples of infrared divergences (IR). These divergences are related with the long distance behavior of QCD, but at the same time they play an important role in the short distance behavior of the DY process. The IR divergence emerges due to the presence of a massless field, in our case the gluon. If this massless field couples to another massless field, like the quarks that we are considering here, or to itself, a second type of IR divergence appears, which is called collinear (CO) divergence.

So we have two types IR divergences:

- Soft divergence $m_G \to 0$
- Collinear divergence $m_q \to 0$

Here, m_G and m_q are fictitious masses for the gluon and quark respectively. Since both types appear in the massless limit, they are also generically called mass divergences [98].

Let us see how this appears in a generic diagram with gluon emission:



p_i represents either p_A or p_B, the 4-momentum of the incoming quark or antiquark, and k is the 4-momentum of the radiated gluon. Let us remember that

$$\begin{aligned} p_i &= \xi_i \left(\frac{\sqrt{S}}{2}, 0, 0, \frac{\sqrt{S}}{2} \right) \\ k &= (k_0, k_T, 0, k_z) \end{aligned}$$

so

$$p_i \cdot k = \xi_i \frac{\sqrt{S}}{2} k_0 \left(1 - \cos\theta_{ik}\right) \tag{4.11}$$

where we have used the relation $k^2 = k_0^2 - k_T^2 - k_z^2 = 0$. We can now identify when $p_i \cdot k \to 0$:

- $k_0 \to 0$;
- $\cos\theta_{ik} \to 1$ i.e when $\theta_{ik} \to 0$

In the second case, for instance, the radiated gluon becomes collinear with the incoming parton, forcing the adjacent propagator, which is proportional to $1/(p_i - k)^2 = 1/(-2p_i \cdot k)$, to become singular. Notice that this is only possible because we are assuming $p_i^2 = 0$, i.e a massless quark. For the first case, we have a soft gluon, since $\vec{k} \to 0$, and as consequence we also have [2] $k_T \to 0$. Therefore, at the same time that the gluon goes soft also becomes collinear. There is a superposition of the soft and collinear singularities.

The role of these singularities can be better appreciated in the integrated cross section. Using Eq.(3.105):

$$h_{q\bar{q}}^{\mu\nu(R)}(-g_{\mu\nu}) = \frac{4e_j^2 g^2}{9} \frac{2}{ut} \left[(Q^2 - t)^2 + (Q^2 - u)^2 \right]$$

[2]Because the gluon is massless.

and since

$$\begin{aligned}\hat{t} &= (k-p_B)^2 = -2p_B\cdot k\\ \hat{u} &= (k-p_A)^2 = -2p_A\cdot k\end{aligned}$$

we can rewrite Eq.(3.105) in the low Q_T limit, in the region $z_A \to 1$ with $\xi_B - x_B$ different from zero [54]:

$$\begin{aligned}&\lim_{Q_T\to 0} h_{q\bar{q}}^{\mu\nu(R)}(-g_{\mu\nu})\\ &\quad = \frac{4e_j^2 g^2}{9}\frac{2}{\hat{u}}\frac{z_B Q^2}{z_B^2}\left[\frac{1+z_B^2}{z_B-1}\right] + \text{finite part}\\ &\quad = \frac{4e_j^2 g^2}{9}\frac{Q^2}{z_B\,(p_A\cdot k)}\left[\frac{1+z_B^2}{1-z_B}\right]\end{aligned}$$

which explicitly shows the divergence associated with gluon emission [54]. Now, including the normalization factors and the delta function, but omitting the parton flux factor, we have from equations (2.18), (2.24) and (3.87):

$$\frac{d\sigma}{d^4q} = \frac{\alpha^2}{12S^2Q^2\pi^3}\frac{4e_j^2g^2}{9}4\pi\frac{S}{Q_T^2}\left[\frac{1+z_B^2}{1-z_B}\right]\delta\left(1-z_A\right) \tag{4.12}$$

$$\frac{d\sigma}{d^4q} = \frac{4}{9}\frac{\alpha^2}{SQ^2}\frac{\alpha_S(Q)}{\pi}\frac{4}{3}\frac{1}{Q_T^2}\left[\frac{1+z_B^2}{1-z_B}\right]\delta\left(1-z_A\right) \tag{4.13}$$

$$\frac{d\sigma}{dQ^2dydQ_T^2} = e_j^2\sigma_0\cdot\frac{\gamma_{qq}}{Q_T^2}\,\delta\left(1-z_A\right) \tag{4.14}$$

where we have defined:

$$\begin{aligned}\sigma_0 &= \frac{4}{9}\frac{\alpha^2\pi}{SQ^2}\\ \gamma_{qq} &= \frac{\alpha_S(Q)}{2\pi}\frac{4}{3}\left[\frac{1+z_B^2}{1-z_B}\right]\end{aligned}$$

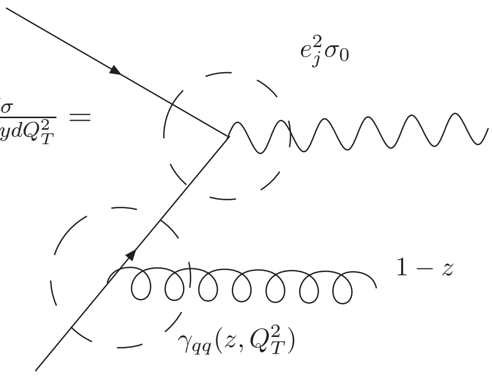
Integrating with respect to Q_T^2 we obtain:

$$\frac{d\sigma}{dQ^2dy} = e_j^2\sigma_0\cdot\gamma_{qq}\,\delta\left(1-z_A\right)\ln\left(\frac{\mu_F^2}{m^2}\right) \tag{4.15}$$

with $\mu_F^2 \approx Q^2$ and $m^2 \approx q_T^2$, where q_T^2 is a sort of minimum transverse momentum chosen as IR cutoff. The logarithm is an explicit consequence of the presence of a collinear divergence.

In the relation (4.14) it is explicitly seen that the cross section can be written as the product of two factors: the first term σ_0 is the total cross section for the DY process at parton level, while the second factor γ_{qq} can be interpreted as the probability of a quark to radiate a gluon and so to become a quark with momentum fraction z and transverse momentum Q_T [78]. This factorization is a typical example of collinear factorization. We have found that in the low Q_T limit[3], the divergent part[4] of the cross section or of the transverse structure function, can be written as a product of two probabilities: the probability of interaction $e_j^2\sigma_0$ and the probability of gluon emission γ_{qq} [78]. These probabilities can be calculated separately and then multiplied. This is the realization of the parton model picture of the DY process in QCD. Pictorially,

$$\frac{d\sigma}{dQ^2dydQ_T^2} =$$



4.3 Resummation

Let us first study the usual formula of the cross section [44], [57] for the DY process with measured Q_T [5]:

$$\frac{d\sigma}{dQ^2dyd^2\vec{Q}_T} = \frac{4\pi\alpha^2}{9SQ^2}\sum_{a,b}\int_{x_A}^1 \frac{d\xi_A}{\xi_A}\int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{a/A}(\xi_A,\mu)\, f_{b/B}(\xi_B,\mu)\, T_{ab}\left(Q_T, Q, \frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}; \mu, g(\mu)\right) \tag{4.16}$$

this formula is valid up to corrections m/Q and when $Q_T \approx Q$. The sum runs over all species a and b of partons (i.e gluons and flavors of quarks and antiquarks). The hard scattering function T but not f has a perturbative expansion in powers of $\alpha_S(\mu)$. As we have already

[3] i.e. when the gluons are emitted with low transverse momentum

[4] To be precise, the part proportional to $1/Q_T^2$

[5] Note that the angular dependence has been integrated, compare with equations (B.1), (B.2) and (B.3)

said we have chosen $\mu \approx Q$ to avoid the large logarithms $\ln(Q/\mu)$ which otherwise may spoil the low-order perturbative approximation to T. Writing explicitly the perturbative expansion for T:

$$T_{ab}\left(Q_T, Q, \frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}; \mu, \alpha_S(\mu)\right) = \sum_{n=0}^{\infty} \left[\frac{\alpha_S(\mu)}{\pi}\right]^n T_{ab}^n \left(Q_T, Q, \frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}; \mu\right) \tag{4.17}$$

the lowest order corresponds to the parton picture for DY:

$$T_{ab}^0 = e_a^2 \delta_{a\bar{b}} \delta\left(1 - \frac{x_A}{\xi_A}\right) \delta\left(1 - \frac{x_B}{\xi_B}\right) \delta^2\left(\vec{Q}_T\right) \tag{4.18}$$

The general form for the coefficients is known [57]:

$$\begin{aligned}
& T_{ab}^n\left(Q_T, Q, \frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}; \mu\right) \\
= \; & \mathcal{N}_{ab}^n\left(Q_T, Q, \frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}; \mu\right) \delta^2\left(\vec{Q}_T\right) \\
& + \sum_{m=0}^{2n-1} T_{ab}^{n,m}\left(Q_T, Q, \frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}; \mu\right) \left[\frac{\ln^m(Q^2/Q_T^2)}{Q_T^2}\right] \\
& + \mathcal{R}_{ab}^n\left(Q_T, Q, \frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}; \mu\right)
\end{aligned} \tag{4.19}$$

Where we have divided T_{ab}^n in terms according to their behavior when $Q_T \to 0$. "Divergent terms," those proportional to $1/Q_T^2$, $1/Q_T^2 \times$ logs and $\delta^2\left(\vec{Q}_T\right)$ have been taken out and what is left has been included in the "regular terms" function $\mathcal{R}_{ab}^n$. This separation should be understood in the sense of distributions, i.e. the low Q_T limit needs to be taken after the integration with respect to ξ_A and ξ_B [44].

It is easy to observe that the perturbative expansion of T_{ab} is not dominated by successive powers of $\alpha_S^n(\mu)$ but by terms of the form $\alpha_S^n(\mu)\left[\frac{\ln^{2n-1}(Q^2/Q_T^2)}{Q_T^2}\right]$; thus, the logarithms that we encountered for first time in Eq.(4.3) are a generic feature order by order of the perturbative expansion. These terms are potentially big when $Q_T \to 0$. This fact renders the low terms approximation of T useless. How this problem is fixed is the subject of the next two subsections.

4.3.1 From factorized to resummed formula

The reorganization of soft and collinear divergences is the most amazing result that we are going to present in this dissertation. It was first obtained by Collins and Soper in the analysis of back-to-back jets in e^+e^- [41] and [43], and then extended by the same authors together with G. Sterman for the DY process in [44].

Let us look at the formula obtained by them:

$$\begin{aligned}
\frac{d\sigma}{dQ^2 dy dQ_T^2} \approx\ & \frac{4\pi^2\alpha^2}{9Q^2 S}\frac{1}{(2\pi)^2}\int d^2 b e^{i\vec{Q}_T\cdot\vec{b}}\sum_j e_j^2 \\
& \times \sum_a \int_{x_A}^1 \frac{d\xi_A}{\xi_A} f_{a/A}\left(\xi_A; 1/b\right) \sum_b \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{b/B}\left(\xi_B; 1/b\right) \\
& \times \exp\left\{ - \int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\ln\left(\frac{Q^2}{\bar{\mu}^2}\right) A(g(\bar{\mu})) + B(g(\bar{\mu}))\right]\right\} \\
& \times C_{ja}\left(\frac{x_A}{\xi_A}; g(1/b)\right) C_{\bar{j}b}\left(\frac{x_B}{\xi_B}; g(1/b)\right) \\
& + \frac{4\pi^2\alpha^2}{9Q^2 s} Y_f(Q_T; Q, x_A, x_B)
\end{aligned} \tag{4.20}$$

The sum runs over gluons and flavors of quarks and antiquarks. The $f's$ are the same parton distributions mentioned before but evaluated at renormalization scale $\mu = 1/b$. $\vec{b}$ is the dual variable of $\vec{Q}_T$ in the Fourier transform, i.e b is the impact parameter variable dual to the transverse momentum variable Q_T.

The first term in equation (4.20) is dominant in the cross section when $Q_T \ll Q$ and the Y_f term defined as,

$$\begin{aligned}
& Y_f(Q_T; Q, x_A, x_B) \\
& = \sum_{a,b} \int_{x_A}^1 \frac{d\xi_A}{\xi_A} f_{a/A}\left(\xi_A; \mu\right) \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{b/B}\left(\xi_B; \mu\right) \sum_{n=1}^{\infty}\left(\frac{\alpha_S(\mu)}{\pi}\right)^n \mathcal{R}_{ab}^n\left(Q_T, Q, \frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}; \mu\right)
\end{aligned} \tag{4.21}$$

which becomes important when $Q_T \approx Q$ [44]. The A, B and C functions are calculable in pQCD and the low order coefficients of their expansions in α_S are known [44], [63] and [123].

In order to justify the resummed formula (4.20), we need to start with the properties of the amplitude or the cross section in the "elastic limit" [52], [114]. This limit [6] is characterized by

[6] The reader should observe that the final state in this elastic limit includes an arbitrary number of particles.

a single hard scale Q and a fixed number of jet functions of mass m, negligible compared with the hard scale.

In the elastic limit, a new scale m is introduced with $Q \gg m \gg \Lambda_{QCD}$. The presence of the second scale has as consequence that the perturbative calculation receives logarithmic enhancements in the ratio Q/m for every order in α_S [119].

Resummation of two-scale logarithms can be deduced when a cross section or amplitude is a product or convolution of factors that separate the distinct scales, Q and m, through the introduction of a third scale, the factorization scale $\mu_F \gg \Lambda_{QCD}$. Formally,

$$\sigma(Q, m) = w(Q, \mu_F) \otimes f(\mu_F, m) \tag{4.22}$$

Where there is factorization, there is evolution [119]. Since the physical cross section cannot depend on the factorization scale, any changes in the short distance function w, due to μ_F, must be compensated by changes in the long distance function f,

$$\mu_F \frac{d}{d\mu_F} \ln \sigma(Q, m) = 0 \ \Rightarrow \mu_F \frac{d \ln f}{d\mu_F} = -P\left(\alpha_S\left(\mu_F\right)\right) = -\mu_F \frac{d \ln w}{d\mu_F}$$

where the kernel P can depend only on the variables that the functions hold in common. Where there is evolution, there is resummation , which can be understood as the solution to the evolution equations [52].

In the case of the inclusive Drell-Yan cross section,

$$\mathcal{H}_{a\bar{a}} \otimes \mathcal{P}_{a/A} \otimes \mathcal{P}_{\bar{a}/B} \otimes \mathcal{S}$$

The convolution here is in transverse momenta. The functions $\mathcal{P}_{a/A}$ and $\mathcal{P}_{\bar{a}/B}$ represent the contributions of the two jets, $\mathcal{S}$ represents the contributions of soft quanta not part of the jets and $\mathcal{H}$ that of the hard quanta [52].

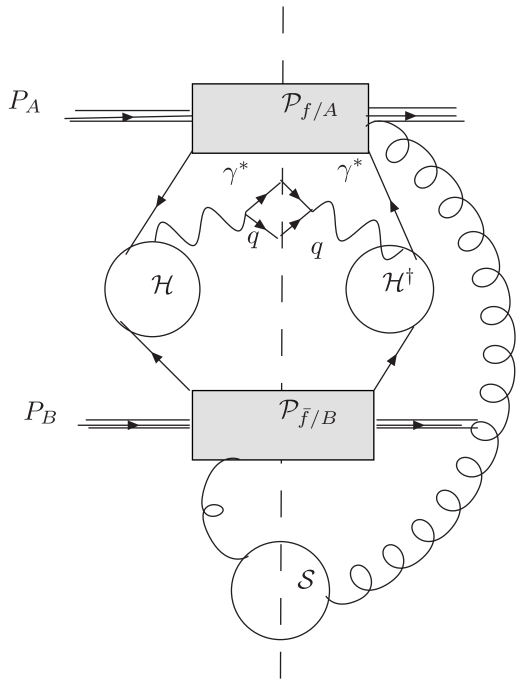
As we have remarked before the DY cross section is not infrared safe but the long distance physics only is present in the jet functions and the soft radiation function $\mathcal{S}$.

By now it should be clear that the low Q_T limit taken in Sec. (4.1) corresponds to the elastic limit of the DY process and equations (4.3) and (4.14) are particular instances of the factorization behavior just described .

Explicitly, in the low Q_T limit for DY with measured pair mass squared Q^2, transverse momentum $\vec{Q_T}$ and rapidity y we have [116]:

$$\frac{d\sigma_{AB}}{dQ^2 dy d^2\vec{Q}_T} = \frac{4\pi\alpha^2}{9Q^2 S}\sum_a \int \frac{d\xi_A}{\xi_A}\int \frac{d\xi_B}{\xi_B}\int \frac{d^2k_T d^2k_T^{'} d^2k_{T,s}}{(2\pi)^6}\delta^2\left(\vec{Q}_T - \vec{k}_T - \vec{k}_T^{'} - \vec{k}_{T,s}\right)$$
$$\times \mathcal{P}_{a/A}\left(\xi_A, k_T; \mu_F\right)\mathcal{P}_{\bar{a}/B}\left(\xi_B, k_T^{'}; \mu_F\right)\mathcal{H}_{a\bar{a}}\left(Q^2; \mu_F\right)\mathcal{S}\left(k_{T,s}; \mu_F\right) \tag{4.23}$$

Note that the collinear factorization holds as convolution in terms of the transverse momenta of gluons emitted from the parton distribution functions associated with the incoming hadrons, along with "central" soft gluons from the soft subdiagram. Symbolically,



In last figure all lines are on-shell and massless, and the virtual photon γ^* with momentum q, $q^2 = Q^2$, is linked to partons through the two hard-scattering functions $\mathcal{H}$ and $\mathcal{H}^\dagger$ in the amplitude and complex conjugate respectively. $\mathcal{H}$ and $\mathcal{H}^\dagger$ depend only on quanta off-shell by order Q^2. On-shell particles with momenta of order Q fall into the two jet functions $\mathcal{P}_{a/A}$ and $\mathcal{P}_{\bar{a}/B}$ of collinear particles [7]. These jet functions have their origin with the incoming partons, so they can be identified as parton distributions [52].

Using

$$\delta^2\left(\vec{Q}_T - \vec{k}_T - \vec{k}_T' - \vec{k}_{T,s}\right) = \int \frac{d^2\vec{b}}{(2\pi)^2} e^{i\vec{b}\cdot\left(\vec{Q}_T - \vec{k}_T - \vec{k}_T' - \vec{k}_{T,s}\right)} \tag{4.24}$$

we can “Fourier transform” Eq.(4.23) to obtain in the elastic limit:

$$\frac{d\sigma_{AB}}{dQ^2 dy d^2\vec{Q}_T} \approx \frac{4\pi\alpha^2}{9Q^2 S}\int \frac{d^2\vec{b}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{Q}_T}\tilde{W}\left(b; Q, x_A, x_B,\right) \tag{4.25}$$

where we have defined :

$$\begin{aligned}
&\tilde{W}\left(b; Q, x_A, x_B,\right)\\
&\quad\equiv \sum_a \int \frac{d\xi_A}{\xi_A}\int \frac{d\xi_B}{\xi_B}\mathcal{H}_{a\bar{a}}\left(\xi_A, \xi_B, Q^2, P_A^+, P_B^-; g\left(\mu_F\right)\right)\\
&\quad\times\tilde{\mathcal{P}}_{a/A}\left(\xi_A, \frac{P_A^+}{\mu_F}, bm; g\left(\mu_F\right)\right)\tilde{\mathcal{P}}_{\bar{a}/B}\left(\xi_B, \frac{P_B^-}{\mu_F}, bm; g\left(\mu_F\right)\right)\\
&\quad\times\tilde{\mathcal{S}}\left(\xi_A, \xi_B, P_A^+, P_B^-, bm; g\left(\mu_F\right)\right)
\end{aligned} \tag{4.26}$$

with

$$\tilde{\mathcal{P}}_{a/A}\left(\xi_A, \frac{P_A^+}{\mu_F}, bm; g\left(\mu_F\right)\right) = \int \frac{d^2\vec{k}_T}{(2\pi)^2} e^{-i\vec{b}\cdot\vec{k}_T}\mathcal{P}_{a/A}\left(\xi_A, k_T; \mu_F\right) \tag{4.27}$$

$$\tilde{\mathcal{P}}_{\bar{a}/B}\left(\xi_B, \frac{P_B^-}{\mu_F}, bm; g\left(\mu_F\right)\right) = \int \frac{d^2\vec{k}_T'}{(2\pi)^2} e^{-i\vec{b}\cdot\vec{k}_T'}\mathcal{P}_{\bar{a}/B}\left(\xi_B, k_T'; \mu_F\right) \tag{4.28}$$

$$\tilde{\mathcal{S}}\left(\xi_A, \xi_B, P_A^+, P_B^-, bm; g\left(\mu_F\right)\right) = \int \frac{d^2\vec{k}_{T,s}}{(2\pi)^2} e^{-i\vec{b}\cdot\vec{k}_{T,s}}\mathcal{S}\left(\xi_A, \xi_B, k_{T,s}; \mu_F\right) \tag{4.29}$$

The jet functions are matrix elements of quark fields separated by a spacelike vector $\left(0^+, y^-, \vec{b}\right)$,

$$\tilde{\mathcal{P}}_{a/A}\left(\xi_A, \frac{P_A^+}{\mu_F}, bm; g\left(\mu_F\right)\right) = \frac{1}{4\pi}\int dy^-\, e^{-i\xi_A P_A^+ y^-}\left\langle P_A\left|\bar{\psi}^{(a)}(0^+, y^-, \vec{b})\gamma^+\psi^{(a)}(0)\right| P_A\right\rangle \tag{4.30}$$

where a spin average has been suppressed. These matrix elements are gauge-dependent and in the light-cone gauge or any physical gauge they absorb all double logarithms[8] of b (or Q_T in

[7] This is an equivalent description of the “elastic limit.”

[8] These logarithms come from integrating with respect to Q_T the differential cross section or the transverse structure function, Eq.(4.3)

momentum space) [116]. At $b = 0$, they coincide with the previously defined quark distribution functions (3.59) and (3.60).

Simplifying the notation and exhibiting explicitly the gauge dependence we write:

$$\tilde{W}(b,Q) = \mathcal{H}\left(\frac{p_A\cdot n}{\mu_F},\frac{p_B\cdot n}{\mu_F};g(\mu_F)\right)\tilde{\mathcal{P}}\left(\frac{p_A\cdot n}{\mu_F};g(\mu_F)\right)\tilde{\mathcal{P}}\left(\frac{p_B\cdot n}{\mu_F};g(\mu_F)\right)\tilde{\mathcal{S}}\left(p_A\cdot n,p_B\cdot n;g(\mu_F)\right) \tag{4.31}$$

Since $\tilde{W}(b,Q)$ is independent of the factorization scale,

$$\mu_F\frac{d}{d\mu_F}\ln\tilde{W}(b,Q) = 0$$

we obtain,

$$\begin{aligned}
\mu_F\frac{\partial}{\partial\mu_F}\ln\tilde{\mathcal{H}}\left(\frac{p_A\cdot n}{\mu_F},\frac{p_B\cdot n}{\mu_F};g(\mu_F)\right) &= -\gamma_H\left(g(\mu_F)\right) &(4.32)\\
\mu_F\frac{\partial}{\partial\mu_F}\ln\tilde{\mathcal{P}}\left(\frac{p_i\cdot n}{\mu_F};g(\mu_F)\right) &= -\gamma_i\left(g(\mu_F)\right) &(4.33)\\
\mu_F\frac{\partial}{\partial\mu_F}\ln\tilde{\mathcal{S}}\left(p_A\cdot n,p_B\cdot n;g(\mu_F)\right) &= -\gamma_S\left(g(\mu_F)\right) &(4.34)
\end{aligned}$$

Here the gammas are anomalous dimensions with

$$\gamma_H+\sum_i\gamma_i+\gamma_S = 0 \tag{4.35}$$

which means that generally speaking each of the functions $\tilde{\mathcal{H}}$, $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{S}}$ needs renormalization and we assume it to be multiplicative. Note that their individual renomalization dependence cancels in their product [52].

A similar analysis can be done with the gauge-fixing vector n [116]. $\tilde{W}(b,Q)$ is independent of n. Thus under a variation in $p_i\cdot n$ at fixed n^2 or any other implicit n-dependence we have,

$$(p_A\cdot n)^2\frac{d}{d(p_A\cdot n)^2}\ln\tilde{W}(b,Q) = 0 \tag{4.36}$$

which implies,

$$\begin{aligned} 0 &= (p_A\cdot n)^2\frac{\partial}{\partial(p_A\cdot n)^2}\ln\tilde{\mathcal{H}}\left(\frac{p_A\cdot n}{\mu_F},\frac{p_B\cdot n}{\mu_F};g(\mu_F)\right)\\ &+ (p_A\cdot n)^2\frac{\partial}{\partial(p_A\cdot n)^2}\ln\tilde{\mathcal{P}}\left(\frac{p_A\cdot n}{\mu_F};g(\mu_F)\right)+(p_A\cdot n)^2\frac{\partial}{\partial(p_A\cdot n)^2}\ln\tilde{\mathcal{S}}(p_A\cdot n,p_B\cdot n;g(\mu_F)) \end{aligned}$$

The above equation can be rewritten as,

$$\begin{aligned} &(p_A\cdot n)^2\frac{\partial}{\partial(p_A\cdot n)^2}\ln\tilde{\mathcal{P}}\left(\frac{p_A\cdot n}{\mu_F},mb\,;g(\mu_F)\right)\\ &=\underbrace{-(p_A\cdot n)^2\frac{\partial}{\partial(p_A\cdot n)^2}\ln\tilde{\mathcal{H}}\left(\frac{p_A\cdot n}{\mu_F},\frac{p_B\cdot n}{\mu_F}\,;g(\mu_F)\right)}_{G\left(\frac{p_A\cdot n}{\mu_F};g(\mu_F)\right)}\\ &\underbrace{-(p_A\cdot n)^2\frac{\partial}{\partial(p_A\cdot n)^2}\ln\tilde{\mathcal{S}}(p_A\cdot n,p_B\cdot n,mb\,;g(\mu_F))}_{K(mb;\,g(\mu_F))} \end{aligned} \tag{4.37}$$

where we have introduced the two new functions G and K which match the changes of the hard part and soft part respectively [52]. By Eq.(4.33),

$$(p_A\cdot n)^2\frac{\partial}{\partial(p_A\cdot n)^2}\left(\mu_F\frac{\partial}{\partial\mu_F}\ln\tilde{\mathcal{P}}\right)=0$$

so the combination $G+K$ is renormalization invariant [52],

$$(p_A\cdot n)^2\frac{\partial}{\partial(p_A\cdot n)^2}\gamma_a(g(\mu_F))=\mu_F\frac{\partial}{\partial\mu_F}\left[G\left(\frac{p_A\cdot n}{\mu_F};g(\mu_F)\right)+K(mb;\;g(\mu_F))\right]=0$$

and by separation variables,

$$\begin{aligned} \mu_F\frac{\partial}{\partial\mu_F}K(mb;\;g(\mu_F)) &= -\gamma_K(g(\mu_F))\\ \mu_F\frac{\partial}{\partial\mu_F}G\left(\frac{p_A\cdot n}{\mu_F};g(\mu_F)\right) &= \gamma_K(g(\mu_F)) \end{aligned}$$

Now, integrating the first equation we get

$$K(mb\,;g(\mu_F))=-\frac{1}{2}\int_{m^2}^{\mu_F^2}\frac{d\mu'^2}{\mu'^2}\gamma_K\left(g\left(\mu'^2\right)\right)+K(mb\,;g(m))$$

and choosing [9]

$$\begin{aligned} m &\approx c_1/b\\ \mu_F &\approx c_2 Q \end{aligned} \tag{4.38}$$

[9]Thus $c_1\approx 1$

we find [44]:

$$K\left(mb\,;g\left(\mu_F\right)\right)+G\left(\frac{p_A\cdot n}{\mu_F};g\left(\mu_F\right)\right)$$
$$=-\frac{1}{2}\int_{(c_1/b)^2}^{(c_2Q)^2}\frac{d\mu'^2}{\mu'^2}\,\gamma_K\left(g\left(\mu'^2\right)\right)+K\left(c_1\,;g\left(\frac{c_1}{b}\right)\right)+G\left(\frac{1}{c_2}\,;g\left(c_2Q\right)\right) \tag{4.39}$$

The numbers c_1, c_2 are integration constants that can be used, later on, to compare with perturbative calculations.

We can simplify Eq.(4.39) using the following identity [44]:

$$F(b,Q)=-\int_{(c_1/b)^2}^{(c_2Q)^2}\frac{d\left(1/\bar{b}^2\right)}{1/\bar{b}^2}\frac{\partial}{\partial\ln\left(1/\bar{b}^2\right)}F(\bar{b},Q)+F\left(c_1/(c_2Q),Q\right)$$

Let $F(b,Q)=K\left(mb\,;g\left(\mu_F\right)\right)+G\left(\frac{Q}{\mu_F};g\left(\mu_F\right)\right)$ so,

$$K\left(mb\,;g\left(\mu_F\right)\right)+G\left(\frac{Q}{\mu_F};g\left(\mu_F\right)\right)$$
$$=-\left\{\int_{(c_1/b)^2}^{(c_2Q)^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}A\left(g\left(\bar{\mu}\right)\right)+B\left(c_1,c_2,g(c_2Q)\right)\right\} \tag{4.40}$$

with

$$\begin{aligned}
A\left(g\left(\bar{\mu}\right)\right) &\equiv \frac{1}{2}\frac{\partial K\left(c_1,g\left(\bar{\mu}\right)\right)}{\partial g}\beta\left(g\left(\bar{\mu}\right)\right)+\frac{1}{2}\gamma_K\left(g\left(\bar{\mu}\right)\right)\\
B\left(c_1,c_2,g(c_2Q)\right) &\equiv -K\left(c_1,g(c_2Q)\right)-G\left(1/c_2,g(c_2Q)\right)
\end{aligned}$$

Going back to Eq.(4.37) we can write, using Eq.(4.40):

$$\left(p_A\cdot n\right)^2\frac{\partial}{\partial\left(p_A\cdot n\right)^2}\ln\tilde{\mathcal{P}}\left(\frac{p_A\cdot n}{\mu_F},c_1\,;g\left(\mu_F\right)\right)=-\left\{\int_{(c_1/b)^2}^{(c_2Q)^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}A\left(g\left(\bar{\mu}\right)\right)+B\left(c_1,c_2,g(c_2Q)\right)\right\} \tag{4.41}$$

and integrating,

$$\ln\tilde{\mathcal{P}}\left(\frac{1}{c_2},c_1\,;g\left(c_2Q\right)\right)=-\int_{(c_1/b)}^{(c_2Q)}d\ln\left(\bar{Q}^2\right)\left[\int_{(c_1/b)^2}^{(c_2\bar{Q})^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}A\left(g\left(\bar{\mu}\right)\right)+B\left(c_1,c_2,g(c_2\bar{Q})\right)\right] \tag{4.42}$$

Here we have employed the following relation[10]

$$p_A\cdot n=p_A^+=\xi_A P_A^+=\xi_A\sqrt{\frac{S}{2}}\rightarrow\frac{Q}{\sqrt{2}}\,e^y$$

[10]Therefore $\left(p_A\cdot n\right)/\mu_F\approx 1/c_2$ and like c_1, we have $c_2\approx 1$

which is valid when $\xi_A \to x_A$, i.e valid in the elastic limit.

Eq.(4.42) can be rewriten as,

$$\begin{aligned}
\ln \tilde{\mathcal{P}}\left(\frac{1}{c_2}, c_1\,; g\left(c_2 Q\right)\right) &= -\int_{(c_1/b)}^{(c_2 Q)} \frac{d\bar{Q}^2}{\bar{Q}^2}\left[\int_{(c_1/b)^2}^{(c_2\bar{Q})^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} A\left(g\left(\bar{\mu}\right)\right) + B\left(c_1, c_2, g(c_2\bar{Q})\right)\right] \\
&= -\int_{(c_1/b)^2}^{(c_2 Q)^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\int_{(\bar{\mu})^2}^{(c_2\bar{Q})^2} \frac{d\bar{Q}^2}{\bar{Q}^2} A\left(g\left(\bar{\mu}\right)\right) + B\left(c_1, c_2, g(c_2\bar{\mu})\right)\right] \\
&= -\int_{(c_1/b)^2}^{(c_2 Q)^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[A\left(g\left(\bar{\mu}\right)\right)\ln\left(\frac{c_2^2 Q^2}{\bar{\mu}^2}\right) + B\left(c_1, c_2, g(c_2\bar{\mu})\right)\right]
\end{aligned} \tag{4.43}$$

We are almost there. To solve equations (4.33) and (4.41) we write,

$$\begin{aligned}
&\tilde{\mathcal{P}}_{a/A}\left(\xi_A, \frac{P_A^+}{\mu_F}, bm; g\left(\mu_F\right)\right) \\
&\quad = \exp\left\{-\int_{(c_1/b)^2}^{(c_2 Q)^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[A\left(g\left(\bar{\mu}\right)\right)\ln\left(\frac{c_2^2 Q^2}{\bar{\mu}^2}\right) + B\left(c_1, c_2, g(c_2\bar{\mu})\right)\right]\right\} \\
&\quad\quad \times \exp\left\{-\int_{\mu_R}^{\mu_F} \frac{d\mu'}{\mu'}\gamma_a\left(\mu'\right)\right\}\tilde{\mathcal{P}}_{a/A}\left(\xi_A, Qb, c_1;\ g\left(\mu_R\right)\right)
\end{aligned} \tag{4.44}$$

Here we have introduced μ_R, an arbitrary renormalization scale, which is also the third integration constant. Like the first two, it will be used also to match the resummed formula with the perturbative calculation. We will define,

$$\mu_R = m = \frac{c_3}{b} \tag{4.45}$$

and choose the values of c_1, c_2 and c_3 according to [7] (also [44]):

$$c_1 = c_3 = 2e^{-\gamma_E}, \quad c_2 = 1, \quad \text{where } \gamma_E \text{ is Euler's constant,} \tag{4.46}$$

or we could have followed the most elaborated analysis of [63] where the integration limits in the Sudakov factor are used to match the NLO results of [2].

Solving equations (4.22) and (4.36) we find for $\tilde{W}(b,Q)$,

$$\begin{aligned}
&\tilde{W}(b,Q)\\
&\quad = \exp\left\{-\int_{(c_1/b)^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[A\left(g\left(\bar{\mu}\right)\right)\ln\left(\frac{Q^2}{\bar{\mu}^2}\right)+B\left(c_1,g(\bar{\mu})\right)\right]\right\}\\
&\quad \times\tilde{\mathcal{P}}_{a/A}\left(\xi_A,Qb,c_1;\ g\left(c_3/b\right)\right)\tilde{\mathcal{P}}_{\bar{a}/A}\left(\xi_B,Qb,c_1;\ g\left(c_3/b\right)\right)\\
&\quad \times\mathcal{H}\left(\frac{x_A}{\xi_A},\frac{x_B}{\xi_B},Qb;\ g\left(c_3/b\right),c_3/b\right)\tilde{\mathcal{S}}\left(c_1;g\left(c_3/b\right)\right) \qquad (4.47)
\end{aligned}$$

where equations (4.35) and (4.44) were used.

At this moment, we will assume that in the collinear configuration the x_A/ξ_A and a dependence factorizes from the x_B/ξ_B and b dependence[11]:

$$\begin{aligned}
&\tilde{W}(b;Q,x_A,x_B)\\
&\quad = \sum_{a,b}\int_{x_A}^{1}\frac{d\xi_A}{\xi_A}\int_{x_B}^{1}\frac{d\xi_B}{\xi_B}f_{a/A}\left(\xi_A;c_1/b\right)f_{b/B}\left(\xi_B;c1/b\right)\\
&\quad \times\sum_j e_j^2 C_{ja}\left(\frac{x_A}{\xi_A},b;\frac{c_1}{c_2};g(c1/b),c1/b\right)C_{\bar{j}a}\left(\frac{x_B}{\xi_B},b;\frac{c_1}{c_2};g(c1/b),c1/b\right) \qquad (4.48)
\end{aligned}$$

here $j=u,\bar{u},d,\bar{d},\ldots$ is the flavor of the annihilating quark or antiquark from hadron A and as has been usual, e_j is its charge in units of e [44].

So we have found for $\tilde{W}(b;Q,x_A,x_B)$ [44],

$$\begin{aligned}
&\tilde{W}(b;Q,x_A,x_B)\\
&\quad = \sum_j e_j^2\sum_a\int_{x_A}^{1}\frac{d\xi_A}{\xi_A}f_{a/A}\left(\xi_A;c_1/b\right)\sum_b\int_{x_B}^{1}\frac{d\xi_B}{\xi_B}f_{b/B}\left(\xi_B;c_1/b\right)\\
&\quad \times\exp\left\{-\int_{(c_1/b)^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[A\left(g\left(\bar{\mu}\right)\right)\ln\left(\frac{Q^2}{\bar{\mu}^2}\right)+B\left(c_1,g(\bar{\mu})\right)\right]\right\}\\
&\quad \times C_{ja}\left(\frac{x_A}{\xi_A},b;\frac{c_1}{c_2};g(c_1/b),c_1/b\right)C_{\bar{j}a}\left(\frac{x_B}{\xi_B},b;\frac{c_1}{c_2};g(c_1/b),c_1/b\right) \qquad (4.49)
\end{aligned}$$

The functions C_i are known as coefficient functions which transform the parton distribution functions f into distributions $C_i\otimes f_i$ specific of the process at hand.

[11]This assumption can be justified assuming that partons a and $\bar{a}$ radiate independently, i.e their respective fields do not overlap

The final step is to plug in the result of Eq.(4.49) into Eq.(4.25) to obtain the resummed formula for the cross section at low Q_T:

$$\begin{aligned}
&\frac{d\sigma}{dQ^2 dy d^2\vec{Q}_T} \\
&\quad \approx \frac{4\pi\alpha^2}{9Q^2 S}\int \frac{d^2\vec{b}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{Q}_T}\tilde{W}\left(b; Q, x_A, x_B\right) \\
&\quad = \frac{4\pi\alpha^2}{9Q^2 S}\int \frac{d^2\vec{b}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{Q}_T}\sum_j e_j^2 \sum_a \int_{x_A}^1 \frac{d\xi_A}{\xi_A} f_{a/A}\left(\xi_A; c_1/b\right) \\
&\quad \times \sum_b \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{b/B}\left(\xi_B; c_1/b\right) \\
&\quad \times \exp\left\{-\int_{(c_1/b)^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[A\left(g\left(\bar{\mu}\right)\right)\ln\left(\frac{Q^2}{\bar{\mu}^2}\right) + B\left(c_1, g(\bar{\mu})\right)\right]\right\} \\
&\quad \times C_{ja}\left(\frac{x_A}{\xi_A}, b; \frac{c_1}{c_2}; g(c_1/b), c_1/b\right) C_{\bar{j}a}\left(\frac{x_B}{\xi_B}, b; \frac{c_1}{c_2}; g(c_1/b), c_1/b\right)
\end{aligned} \tag{4.50}$$

To have a formula valid also when $Q_T \approx Q$ we will include the Y term defined in Eq.(4.21)

$$\frac{d\sigma}{dQ^2 dy d^2\vec{Q}_T} \approx \frac{4\pi\alpha^2}{9Q^2 S}\int \frac{d^2\vec{b}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{Q}_T}\tilde{W}\left(b; Q, x_A, x_B\right) + \frac{4\pi\alpha^2}{9Q^2 S} Y(Q_T; Q, x_A, x_B) \tag{4.51}$$

We need to evaluate the two dimensional Fourier transform present in Eq.(4.51). This is done as follows[12]

$$\int \frac{d^2\vec{b}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{Q}_T} = \int \frac{db}{(2\pi)} b\, J_0(Q_T b) = \frac{1}{2\pi Q_T^2}\int dr\, r\, J_0(r)$$

with $r = Q_T b$. Thus we can write

$$\begin{aligned}
&\int \frac{d^2\vec{b}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{Q}_T}\tilde{W}\left(b; Q, x_A, x_B\right) \\
&\quad = \frac{1}{2\pi Q_T^2}\int dr\, r\, J_0(r)\, \tilde{W}\left(\frac{r}{Q_T}; Q, x_A, x_B\right)
\end{aligned} \tag{4.52}$$

and assume that

$$\left|\tilde{W}(r)\right| \le \frac{1}{r} \quad \text{for } 0 \le r < \infty \tag{4.53}$$

[12]Here the integral representation of the Bessel function of the first kind [5]:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{ix\cos\theta}$$

was employed.

in order to guarantee the existence of the integral.

Eq.(4.52) can be further simplified employing the following recurrence relation among Bessel functions [5]:

$$\frac{d}{dx}\left[x\,J_0(x)\right] = xJ_1(x)$$

which allows us to integrate by parts,

$$\int_0^\infty dr\; r\; J_0(r)\; \tilde{W}(r) = -\int_0^\infty dr\; r\; J_1(r)\frac{d\tilde{W}(r)}{dr} \tag{4.54}$$

Notice that we are using:

$$r\; J_1(r)\; \tilde{W}(r)\Big|_0^\infty = 0$$

which is guaranteed by the condition (4.53). A second possibility to obtain similar results is to require the exponential to dominate the value of $\tilde{W}(r)$. In this case we need to adopt the following three conditions:

1. The coefficient functions $C_i(r)$ are continuous for $0 < r < \infty$ with the possibility of poles of finite order at $r = 0$ or $r = \infty$.
2. $A(r) > 0$
3. $|B(r)| < A(r)\ln r$ for $0 \leq r < \infty$.

Conditions two and three guarantee that the value of the integral in the exponential is positive for any value of r. They also guarantee that the following two limits:

$$\begin{aligned}\lim_{r\to 0}\tilde{W}(r) &= 0\\ \lim_{r\to\infty}\tilde{W}(r) &= 0\end{aligned}$$

hold, which allows us to ignore the boundary values during the integration by parts.

4.3.2 From resummed formula to fixed order perturbative analysis

In order for the result of Eq.(4.50) to have any predictive power we need to know the functions A, B and C. In practice, this is not possible. The best that we can do is to find the lower terms of their perturbative expansions and hope that higher terms do not dominate the value of $\tilde{W}(r)$. We need to observe here that the function $\tilde{W}(r)$ can be calculated perturbatively only for small r *i.e* $\frac{1}{b} \gg \Lambda_{QCD}$ and an extrapolation to large values of r *i.e* $\frac{1}{b} \ll \Lambda_{QCD}$ requires non-perturbative input. This extrapolation is required in order to complete the Fourier transform in Eq.(4.52). At least four approaches have been proposed to handle the extension to high values of b [120]:

1. Instead of working in b-space we can work in Q_T-space directly. This was done in the original work of [57] and revived by [64] and [83].

2. Artificially prevent b to reach large values by replacing it with a new variable b_* and parametrize the non-perturbative region in terms of a form factor $F_{ij}^{NP}(Q, b, x_A, x_B)$. Where the "freezing" of b at b_* is achieved by

$$b_* = \frac{b}{\sqrt{1 + (b/b_{max})^2}}, \quad b_* < b_{max} \tag{4.55}$$

with the parameter $b_{max} \approx 1/\Lambda_{QCD}$ which separates the perturbative region from the non-perturbative one. The form of the function F^{NP} is still matter of debate. This approach was first introduced by [44] and it has been selected here to perform the numerical analysis. See Sec.5.1 in particular equations (5.11) and (5.12).

3. Qiu and Zhang [106] and [107] proposed that for $b_* > b_{max}$,

$$\tilde{W}\left(b; Q, x_A, x_B\right) = \tilde{W}^{Pert}\left(b_{max}\right) F^{NP}(b; b_{max}) \tag{4.56}$$

Unlike the original CSS formalism, $\tilde{W}\left(Q, b; x_A, x_B\right)$ is not altered and is independent of the non-perturbative parameters when $b < b_{max}$.

4. In order to avoid the singularity present at $1/b = \Lambda_{QCD}$ we can alter the b-space contour integral. This technique was first introduced in threshold resummation [17] and then adopted by [85] and [117] and also by [26].

All these approaches require introduction of new parameters for a quantitative fit.

To find the lower terms of A, B and C we will proceed as follows: we will find a formal expression for the cross section from the resummed formula Eq.(4.50) valid up to a finite order. Then, we will match this expression with the cross section at the same order obtained in perturbation theory. We will assume that this expansion around $Q \gg \Lambda_{QCD}$ is approximately equal to the product of the perturbative expansions of the exponential and the coefficient functions together with the evolution of the distribution functions[13]. This product should be understood as the multiplication term by term of the respective series.

We will start with the coefficient functions[14]:

$$\begin{aligned} & C_{ja}\left(b; \frac{x_A}{\xi_A}, \frac{c_1}{c_2}; g(\mu;Q), \mu\right) \\ &= \sum_{n=0}^{\infty} \left[\frac{\alpha_S\left(\mu;Q\right)}{\pi}\right]^n \tilde{C}_{ja}^{(n)}\left(\frac{x_A}{\xi_A}, b; \frac{c_1}{c_2}; \mu\right) \\ &= \sum_{n=0}^{\infty} \left[\frac{\alpha_S\left(Q\right)}{\pi}\right]^n C_{ja}^{(n)}\left(\frac{x_A}{\xi_A}, b; \frac{c_1}{c_2}; \mu\right) \end{aligned} \tag{4.57}$$

with

$$\begin{aligned} \frac{\alpha_S(\mu;Q)}{\pi} &= \frac{\alpha_S(Q)}{\pi} - \beta_1 \ln\left(\frac{\mu^2}{Q^2}\right)\left[\frac{\alpha_S(Q)}{\pi}\right]^2 + \ldots \\ \beta_1 &= \frac{1}{12}(33 - 2N_f) \quad \text{here } N_f \text{ is the number of flavors} \end{aligned} \tag{4.58}$$

and there is a similar one for $C_{\bar{j}b}$.

Let us find an expansion for the exponential. We will make use of the fact that A, B and α_S have perturbative series around Q:

$$\begin{aligned} &\exp\left\{-\int_{(c_1/b)^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[A\left(g\left(\bar{\mu}\right)\right)\ln\left(\frac{Q^2}{\bar{\mu}^2}\right) + B\left(c_1, g(\bar{\mu})\right)\right]\right\} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left\{-\int_{(c_1/b)^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[A\left(g\left(\bar{\mu}\right)\right)\ln\left(\frac{Q^2}{\bar{\mu}^2}\right) + B\left(c_1, g(\bar{\mu})\right)\right]\right\}^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left\{-\int_{(c_1/b)^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\sum_{l=1}^{\infty}\left[\frac{\alpha_S\left(\bar{\mu};Q\right)}{\pi}\right]^l A^l\left(c_1\right)\ln\left(\frac{Q^2}{\bar{\mu}^2}\right) + \sum_{m=1}^{\infty}\left[\frac{\alpha_S\left(\bar{\mu};Q\right)}{\pi}\right]^m B^m\left(c_1, c_2\right)\right]\right\}^n \end{aligned} \tag{4.59}$$

[13]Notice that the distribution functions are nonpertubative quantities. Therefore, we do not have an expansion for them in terms of $\alpha_S(Q)$

[14]To simplify notation we will write μ for $\mu_R = c_1/b$

Suppose now that we can exchange the order of the sum and integral sign without affecting the final result. Then,

$$\begin{aligned}
&\exp\left\{-\int_{(c_1/b)^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[A\left(g\left(\bar{\mu}\right)\right)\ln\left(\frac{Q^2}{\bar{\mu}^2}\right)+B\left(c_1,g(\bar{\mu})\right)\right]\right\}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{\sum_{l=1}^{\infty}A^l\left(c_1\right)\int_{(c_1/b)^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\frac{\alpha_S\left(\bar{\mu};Q\right)}{\pi}\right]^l\ln\left(\frac{Q^2}{\bar{\mu}^2}\right)\right.\\
&\left.+\sum_{m=1}^{\infty}B^m\left(c_1,c_2\right)\int_{(c_1/b)^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\frac{\alpha_S\left(\bar{\mu};Q\right)}{\pi}\right]^m\right\}^n \qquad (4.60)\\
&=\sum_{n=0}^{\infty}\left[\frac{\alpha_S(Q)}{\pi}\right]^n F^{(n)}\left(c_1,c_2;\frac{c_1}{b};Q\right) \qquad (4.61)
\end{aligned}$$

Here $F^{(n)}\left(c_1,c_2;\frac{c_1}{b};Q\right)$ is defined order by order in $\alpha_S(Q)/\pi$.

Now, we have evolution of the distribution functions:

$$\mu^2\frac{d}{d\mu^2}f_{j/A}\left(x,\mu\right)=\int_x^1\frac{d\xi}{\xi}\sum_k P_{j/k}\left(\frac{x}{\xi}\right)f_{j/A}\left(\xi,\mu\right) \qquad (4.62)$$

with $P_{j/k}(z)$ the DGLAP kernel. This kernel can be expanded in terms of $\alpha_S(\mu)$

$$P_{j/k}(z;\mu)=\sum_{n=1}^{\infty}\left(\frac{\alpha_S(\mu)}{\pi}\right)^n P^{(n)}_{j/k}(z;\mu) \qquad (4.63)$$

and at order $\mathcal{O}(\alpha_S(Q))$ we have for example:

$$\mu^2\frac{d}{d\mu^2}f_{j/A}\left(x,\mu\right)=\frac{\alpha_S(Q)}{2\pi}\int_x^1\frac{d\xi}{\xi}\sum_k P^{(1)}_{j/k}\left(\frac{x}{\xi}\right)f_{j/A}\left(\xi,Q\right) \qquad (4.64)$$

where $P^{(1)}_{j/k}(z)$ defined in Table (B.1). Thus, we obtain for $\tilde{W}$

$$\tilde{W}\left(b;Q,x_A,x_B\right)=\sum_{n=0}^{\infty}\left[\frac{\alpha_S(Q)}{\pi}\right]^n\tilde{W}^{(n)}\left(b;Q,x_A,x_B;\frac{c_1}{c_2};\mu\right) \qquad (4.65)$$

and at lowest order this is equal to:

$$\begin{aligned}
&\tilde{W}^{(0)}\left(x_A,x_B,b;\frac{c_1}{c_2};\mu\right)\\
&=\sum_j e_j^2\sum_{a,b}\int_{x_A}^1\frac{d\xi_A}{\xi_A}f_{a/A}\left(\xi_A;\mu\right)\int_{x_B}^1\frac{d\xi_B}{\xi_B}f_{b/B}\left(\xi_B;\mu\right)C^{(0)}_{ja}\left(\frac{x_A}{\xi_A},b;\frac{c_1}{c_2};\mu\right)C^{(0)}_{\bar{j}b}\left(\frac{x_B}{\xi_B},b;\frac{c_1}{c_2};\mu\right)
\end{aligned} \qquad (4.66)$$

Here we have used $F^{(0)} = 1$. We can also easily find the coefficient of the leading order using the prediction of the parton model. For $C^{(0)}$ we get [44],

$$C^{(0)}_{ja}\left(z, b; \frac{c_1}{c_2}; \mu\right) = \delta_{ja}\delta(z-1) \tag{4.67}$$

$$C^{(0)}_{jg}\left(z, b; \frac{c_1}{c_2}; \mu\right) = 0 \tag{4.68}$$

These results were obtained comparing the Fourier transform of the singular part of equations (4.17), (4.18) with the perturbative expansion of Eq.(4.48).

In general we find:

$$\begin{aligned}
&\tilde{W}^{(n)}\left(x_A, x_B, b; \frac{c_1}{c_2}; Q\right) \\
&= \sum_j e_j^2 \sum_a \int_{x_A}^1 \frac{d\xi_A}{\xi_A} f_{a/A}(\xi_A; \mu) \sum_b \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{b/B}(\xi_B; \mu) \\
&\times \sum_{k=0}^{n} \sum_{l=0}^{k} C^{(l)}_{ja}\left(\frac{x_A}{\xi_A}, b; \frac{c_1}{c_2}; \mu\right) C^{(k-l)}_{\bar{j}b}\left(\frac{x_B}{\xi_B}, b; \frac{c_1}{c_2}; \mu\right) F^{(n-k)}\left(c_1, c_2; \frac{c_1}{b}; Q\right)
\end{aligned} \tag{4.69}$$

Now we want to use our previous results to find the expansion of $\mathcal{O}(\alpha_S(Q))$ for the cross section deduced from the resummed formula. Therefore, we need to go back to Eq.(4.54). The derivative is equal to

$$-\int_0^\infty dr\; r\; J_1(r) \frac{d\tilde{W}(r)}{dr} = \sum_{n=0}^{\infty} \left[\frac{\alpha_S(Q)}{\pi}\right]^n \left\{-\int_0^\infty dr\; r\; J_1(r) \frac{d\tilde{W}^n(r)}{dr}\right\}$$

which expanded at $\mathcal{O}(\alpha_S(Q))$ is

$$\begin{aligned}
&-\int_0^\infty dr\; r\; J_1(r) \frac{d\tilde{W}^1}{dr} \\
&= -\left(\frac{\alpha_S(Q)}{\pi}\right) \int_0^\infty dr\; r\; J_1(r) \frac{d}{dr}\left\{f_{a/A}\left(\xi_A; \frac{c_1 Q_T}{r}\right) f_{b/B}\left(\xi_B; \frac{c_1 Q_T}{r}\right)\right. \\
&\times \left.\left[C^{(0)}_{ja}(r) C^{(0)}_{\bar{j}b}(r) F^{(1)}(r) + C^{(0)}_{ja}(r) C^{(1)}_{\bar{j}b}(r) + C^{(1)}_{ja}(r) C^{(0)}_{\bar{j}b}(r)\right]\right\}
\end{aligned} \tag{4.70}$$

where we have omitted some integrals and sum factors. Through our particular choice of renormalization constant $\mu \approx c_3/b$ we can make the C's independent of r, which means:

$$
\begin{aligned}
&-\int_0^\infty dr\ r\ J_1(r)\frac{d\tilde{W}^1}{dr} \\
&\quad = -\left(\frac{\alpha_S(Q)}{\pi}\right)\int_0^\infty dr\ r\ J_1(r) \\
&\qquad \times\left\{\left[f_{b/B}\left(\xi_B;\frac{c_1Q_T}{r}\right)\frac{d}{dr}f_{a/A}\left(\xi_A;\frac{c_1Q_T}{r}\right)+f_{a/A}\left(\xi_A;\frac{c_1Q_T}{r}\right)\frac{d}{dr}f_{b/B}\left(\xi_B;\frac{c_1Q_T}{r}\right)\right]\right. \\
&\qquad \times\left[C^{(0)}_{ja}(r)C^{(0)}_{\bar{j}b}(r)F^{(1)}(r)+C^{(0)}_{ja}(r)C^{(1)}_{\bar{j}b}(r)+C^{(1)}_{ja}(r)C^{(0)}_{\bar{j}b}(r)\right] \\
&\qquad \left.+\left[f_{a/A}\left(\xi_A;\frac{c_1Q_T}{r}\right)f_{b/B}\left(\xi_B;\frac{c_1Q_T}{r}\right)\right]\left[C^{(0)}_{ja}C^{(0)}_{\bar{j}b}\frac{d}{dr}F^{(1)}(r)\right]\right\}
\end{aligned}
\tag{4.71}
$$

The terms containing the derivatives of the distribution functions will give a contribution proportional to $\alpha_S^2(Q)$ by Eq.(4.64) and therefore they do not contribute to $\mathcal{O}(\alpha_S(Q))$. So we get:

$$
-\int_0^\infty dr\ r\ J_1(r)\frac{d\tilde{W}^1}{dr} = -\left(\frac{\alpha_S(Q)}{\pi}\right)\int_0^\infty dx\ r\ J_1(r)C^{(0)}_{ja}C^{(0)}_{\bar{j}b}\frac{d}{dr}F^{(1)}(r)+\mathcal{O}(\alpha_S^2(Q)) \tag{4.72}
$$

We are still one term short. We need to take into account a term proportional to $\mathcal{O}(\alpha_S(Q))$ coming from $\tilde{W}^0$:

$$
\begin{aligned}
&-\int_0^\infty dr\ r\ J_1(r)\frac{d\tilde{W}^0}{dr} \\
&\quad = -\int_0^\infty dr\ r\ J_1(r)C^{(0)}_{ja}C^{(0)}_{\bar{j}b}\left[f_{b/B}\left(\frac{1}{r}\right)\frac{d}{dr}f_{a/A}\left(\frac{1}{r}\right)+f_{a/A}\left(\frac{1}{r}\right)\frac{d}{dr}f_{b/B}\left(\frac{1}{r}\right)\right]
\end{aligned}
\tag{4.73}
$$

collecting terms (4.72) and (4.73) we finally obtain:

$$
\begin{aligned}
&-C^{(0)}_{ja}C^{(0)}_{\bar{j}b}\int_0^\infty dr\ r\ J_1(r) \\
&\qquad \times\left[f_{b/B}\left(\frac{1}{r}\right)\frac{d}{dr}f_{a/A}\left(\frac{1}{r}\right)+f_{a/A}\left(\frac{1}{r}\right)\frac{d}{dr}f_{b/B}\left(\frac{1}{r}\right)+\left(\frac{\alpha_S(Q)}{\pi}\right)\frac{d}{dr}F^{(1)}\right]
\end{aligned}
\tag{4.74}
$$

We have now to evaluate the above integral to order $\alpha_S(Q)$. Let us start with the derivatives, both are easy. Using the evolution Eq.(4.64) we can write to order $\alpha_S(Q)$:

$$
\frac{d}{dr}f_{a/A}\left(x;\frac{1}{r}\right) = \frac{-2}{r}\frac{\alpha_S(Q)}{2\pi}\int_x^1\frac{d\xi}{\xi}\sum_k P^{(1)}_{a/k}\left(\frac{x}{\xi}\right)f_{a/A}\left(\xi;Q\right) \tag{4.75}
$$

and likewise for the second derivative,

$$\begin{aligned}
&\frac{d}{dr}F^{(1)}(r) \\
&\quad = -\frac{d}{dr}\left\{\frac{A^{(1)}}{2}\ln^2\left(\frac{Q^2r^2}{Q_T^2c_1^2}\right) + B^{(1)}\ln\left(\frac{Q^2r^2}{Q_T^2c_1^2}\right)\right\} \\
&\quad = -\frac{2}{r}\left\{A^{(1)}\ln\left(\frac{Q^2r^2}{Q_T^2c_1^2}\right) + B^{(1)}\right\} \qquad (4.76)
\end{aligned}$$

Inserting equations (4.75) and (4.76) into (4.74) we find using $f\left(\xi;\frac{1}{r}\right) \approx f\left(\xi;Q\right)$ and equations (4.67), (4.68):

$$\begin{aligned}
&\frac{\alpha_S(Q)}{\pi}\delta_{ja}\delta_{\bar{j}b}\delta(1-z_A)\delta(1-z_B)\int_0^\infty dr\ J_1(r) \\
&\quad\times\Bigg[f_{b/B}\left(\xi_B;Q\right)\int_{x_A}^1\frac{d\xi}{\xi}\sum_k P_{a/k}^{(1)}\left(\frac{x}{\xi}\right)f_{a/A}\left(\xi;Q\right) + f_{a/A}\left(\xi_A;Q\right)\int_{x_B}^1\frac{d\xi}{\xi}\sum_k P_{b/k}^{(1)}\left(\frac{x}{\xi}\right)f_{b/B}\left(\xi;Q\right) \\
&\qquad + f_{a/A}\left(\xi_A;Q\right)f_{b/B}\left(\xi_B;Q\right)\left\{2A^{(1)}\ln\left(\frac{Q^2r^2}{Q_T^2c_1^2}\right) + 2B^{(1)}\right\}\Bigg] \qquad (4.77)
\end{aligned}$$

In Eq.(4.70) we omitted some integral factors and sums which can be evaluated with the deltas to obtain:

$$\begin{aligned}
&\frac{\alpha_S(Q)}{\pi}\sum_j e_j^2\Bigg[f_{\bar{j}/B}\left(x_B;Q\right)\int_{x_A}^1\frac{d\xi_A}{\xi_A}\sum_k P_{j/k}^{(1)}\left(\frac{x_A}{\xi_A}\right)f_{j/A}\left(\xi_A;Q\right) \\
&\qquad + f_{j/A}\left(x_A;Q\right)\int_{x_B}^1\frac{d\xi_B}{\xi_B}\sum_k P_{\bar{j}/k}^{(1)}\left(\frac{x_B}{\xi_B}\right)f_{\bar{j}/B}\left(\xi_B;Q\right) \\
&\qquad + f_{j/A}\left(x_A;Q\right)f_{\bar{j}/B}\left(x_B;Q\right)\left\{2A^{(1)}\ln\left(\frac{Q^2}{Q_T^2}\right) + 2B^{(1)}\right\}\Bigg] \qquad (4.78)
\end{aligned}$$

here the following two integrals where employed [72]:

$$\int_0^\infty dr\ J_1(r) = 1 \qquad (4.79)$$

$$\int_0^\infty dr\ J_1(r)\ln\left(\frac{r}{c_1}\right) = 0 \qquad (4.80)$$

Remember that $c_1 = 2e^{\gamma_E}$. Formula (4.78) allows to write for the differential cross section predicted from the resummed equation at order $\alpha_S(Q)$ when $Q_T \ll Q$

$$\begin{aligned}
&\frac{d\sigma}{dQ^2 dy d^2\vec{Q}_T} \\
&\quad \approx \frac{4\pi\alpha^2}{9Q^2 S}\frac{1}{2\pi Q_T^2}\left(\frac{\alpha_S(Q)}{\pi}\right)\sum_j e_j^2 \left[f_{\bar{j}/B}\left(x_B;Q\right)\int_{x_A}^1 \frac{d\xi_A}{\xi_A}\sum_k P_{j/k}^{(1)}\left(\frac{x_A}{\xi_A}\right) f_{j/A}\left(\xi_A;Q\right)\right. \\
&\qquad +f_{a/A}\left(x_A;Q\right)\int_{x_B}^1 \frac{d\xi_B}{\xi_B}\sum_k P_{\bar{j}/k}^{(1)}\left(\frac{x_B}{\xi_B}\right) f_{\bar{j}/B}\left(\xi_B;Q\right) \\
&\qquad \left. + f_{j/A}\left(x_A;Q\right) f_{\bar{j}/B}\left(x_B;Q\right)\left\{2A^{(1)}\ln\left(\frac{Q^2}{Q_T^2}\right) + 2B^{(1)}\right\}\right] \qquad (4.81)
\end{aligned}$$

4.3.3 A and B functions

We need now the perturbative expression. It can be easily obtained from equations[15] (2.18), (3.12) and (4.3)

$$\begin{aligned}
&\frac{d\sigma}{dQ^2 dy d^2\vec{Q}_T} \\
&\quad \approx \frac{4\pi\alpha^2}{9Q^2 S}\frac{1}{2\pi Q_T^2}\left(\frac{\alpha_S(Q)}{\pi}\right)\sum_j e_j^2 \left[f_{\bar{j}/B}\left(x_B;Q\right)\int_{x_A}^1 \frac{d\xi_A}{\xi_A}\sum_k P_{j/k}^{(1)}\left(\frac{x_A}{\xi_A}\right) f_{j/A}\left(\xi_A;Q\right)\right. \\
&\qquad +f_{j/A}\left(x_A;Q\right)\int_{x_B}^1 \frac{d\xi_B}{\xi_B}\sum_k P_{\bar{j}/k}^{(1)}\left(\frac{x_B}{\xi_B}\right) f_{\bar{j}/B}\left(\xi_B;Q\right) \\
&\qquad \left. +2\ f_{j/A}\left(x_A;Q\right) f_{\bar{j}/B}\left(x_B;Q\right)\left\{\frac{4}{3}\ln\left(\frac{Q^2}{Q_T^2}\right) - 2\right\}\right] \qquad (4.82)
\end{aligned}$$

where we have used the equality:

$$\left(\frac{1+z^2}{1-z}\right)_+ = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \qquad (4.83)$$

See Appendix (C). From equations (4.81) and (4.82) we conclude:

$$A^{(1)} = \frac{4}{3} \quad B^{(1)} = -2 \qquad (4.84)$$

[15] Notice that W_L^R is finite at low Q_T.

For completeness we will include here the known predictions for A and B [13]:

$$
\begin{aligned}
A^{(1)}(c_1) &= C_F \\
A^{(2)}(c_1) &= C_F\left[\left(\frac{67}{36}-\frac{\pi^2}{12}\right)N_C-\frac{5}{18}N_f-2\beta_1\ln\left(\frac{2e^{-\gamma_E}}{c_1}\right)\right] \\
B^{(1)}(c_1,c_2) &= C_F\left[-\frac{3}{2}-2\ln\left(\frac{2e^{-\gamma_E}c_2}{c_1}\right)\right] \\
B^{(2)}(c_1,c_2) &= C_F\left\{C_F\left(\frac{\pi^2}{4}-\frac{3}{16}-3\zeta(3)\right)+N_C\left(\frac{11}{36}\pi^2-\frac{193}{48}+\frac{3}{2}\zeta(3)\right)\right. \\
&\quad +\frac{N_f}{2}\left(-\frac{1}{9}\pi^2+\frac{17}{12}\right)-\left[\left(\frac{67}{18}-\frac{\pi^2}{6}\right)N_C-\frac{5}{9}N_f\right]\ln\left(\frac{2e^{-\gamma_E}c_2}{c_1}\right) \\
&\quad \left.+2\beta_1\left[\ln^2\left(\frac{2e^{-\gamma_E}}{c_1}\right)-\ln^2(c_2)-\frac{3}{2}\ln(c_2)\right]\right\}
\end{aligned}
$$

where N_f is the number of light quark flavors, $C_F=\mathrm{tr}(t_at_a)$ is the second order Casimir of the quark representation (with t_a being the SU(N_C) generators in the fundamental representation), $\beta_1=(11N_C-2N_f)/12$ and $\zeta(x)$ is the Riemann zeta function, and $\zeta(3)\approx 1.202$. For QCD, $N_C=3$ and $C_F=4/3$. Using the canonical selection for the constants c_1, c_2, Eq.(4.46) the second order coefficients in the Sudakov exponent simplify to

$$
A^{(2)}(c_1=2e^{-\gamma_E}) = C_F\left[\left(\frac{67}{36}-\frac{\pi^2}{12}\right)N_C-\frac{5}{18}N_f\right] \tag{4.85}
$$

$$
B^{(2)}(c_1=2e^{-\gamma_E},c_2=1) = C_F^2\left(\frac{\pi^2}{4}-\frac{3}{16}-3\zeta(3)\right)+C_FN_C\left(\frac{11}{36}\pi^2-\frac{193}{48}+\frac{3}{2}\zeta(3)\right) \tag{4.86}
$$

$$
+C_FN_f\left(-\frac{1}{18}\pi^2+\frac{17}{24}\right) \tag{4.87}
$$

4.3.4 C functions

Now we should turn our attention to the C functions. We cannot find their first order coefficients directly from $\tilde{W}^1$, since they only appear at second order[16]. Fortunately there is a quantity of order α_S that can help us. We are going to Fourier transform the divergent part of the cross section,

$$
\int d^2\vec{Q}_T\, e^{-i\vec{b}\cdot\vec{Q}_T}\frac{d\sigma}{dQ^2dyd^2\vec{Q}_T}\approx\frac{4\pi\alpha^2}{9Q^2S}\tilde{W}\left(b;Q,x_A,x_B\right) \tag{4.88}
$$

[16]See equations (4.71) and (4.72).

and expand $\tilde{W}$ around $\mu = \frac{c_1}{b} = Q$. As consequence of this choice, the exponential factor is reduced to unity. Then, we have:

$$\begin{aligned}
&\int d^2\vec{Q}_T\, e^{-i\vec{b}\cdot\vec{Q}_T} \frac{d\sigma}{dQ^2 dy d^2\vec{Q}_T} \\
&\approx \frac{4\pi\alpha^2}{9Q^2 S} \sum_{a,b} \int_{x_A}^1 \frac{d\xi_A}{\xi_A} \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{a/A}(\xi_A; Q)\, f_{b/B}(\xi_B; Q) \\
&\quad \times \sum_j e_j^2 C_{ja}\left(\frac{x_A}{\xi_A}, b; \frac{c_1}{c_2}; g(Q), Q\right) C_{\bar{j}a}\left(\frac{x_B}{\xi_B}, b; \frac{c_1}{c_2}; g(Q), Q\right) \\
&= \frac{4\pi\alpha^2}{9Q^2 S} \sum_{a,b} \int_{x_A}^1 \frac{d\xi_A}{\xi_A} \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{a/A}(\xi_A; Q)\, f_{b/B}(\xi_B; Q) \\
&\quad \times \sum_j e_j^2 \left(\frac{\alpha_S(Q)}{\pi}\right)^n \sum_{k=0}^{n} C_{ja}^{(k)}\left(\frac{x_A}{\xi_A}, b; \frac{c_1}{c_2}; Q\right) C_{\bar{j}b}^{(n-k)}\left(\frac{x_B}{\xi_B}, b; \frac{c_1}{c_2}; Q\right) \qquad (4.89)
\end{aligned}$$

which at $\mathcal{O}(\alpha_S(Q))$ is equal to:

$$\begin{aligned}
&\int d^2\vec{Q}_T\, e^{-i\vec{b}\cdot\vec{Q}_T} \frac{d\sigma^{(1)}}{dQ^2 dy d^2\vec{Q}_T} \\
&\approx \frac{4\pi\alpha^2}{9Q^2 S}\left(\frac{\alpha_S(Q)}{\pi}\right) \sum_{a,b} \int_{x_A}^1 \frac{d\xi_A}{\xi_A} \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{a/A}(\xi_A; Q)\, f_{b/B}(\xi_B; Q) \\
&\quad \times \sum_j e_j^2 \left[C_{ja}^{(1)}\left(\frac{x_A}{\xi_A}, b; \frac{c_1}{c_2}; Q\right) C_{\bar{j}b}^{(0)}\left(\frac{x_B}{\xi_B}, b; \frac{c_1}{c_2}; Q\right) + C_{ja}^{(0)}\left(\frac{x_A}{\xi_A}, b; \frac{c_1}{c_2}; Q\right) C_{\bar{j}b}^{(1)}\left(\frac{x_B}{\xi_B}, b; \frac{c_1}{c_2}; Q\right)\right] \qquad (4.90) \\
&= \frac{4\pi\alpha^2}{9Q^2 S}\left(\frac{\alpha_S(Q)}{\pi}\right) \sum_a \sum_j e_j^2 \left[f_{\bar{j}/B}(x_B; Q) \int_{x_A}^1 \frac{d\xi_A}{\xi_A} f_{a/A}(\xi_A; Q)\, C_{ja}^{(1)}\left(\frac{x_A}{\xi_A}, b; \frac{c_1}{c_2}; Q\right)\right. \\
&\quad \left. + f_{j/A}(x_A; Q) \int_{x_B}^1 \frac{d\xi_B}{\xi_B} f_{a/B}(\xi_B; Q)\, C_{\bar{j}a}^{(1)}\left(\frac{x_B}{\xi_B}, b; \frac{c_1}{c_2}; Q\right)\right] \qquad (4.91)
\end{aligned}$$

In order to calculate the perturbative contribution we need to Fourier transform Eq. (4.82) and since $Q_T = 0$ is in the range of this integral we also need to include the contributions to the cross section at this value. Both terms, the real contribution and the virtual contribution require regularization; we will use dimensional regularization with dimension $n = 2 - 2\epsilon$

Let us start with the real contribution. We have to modify Eq. (4.82) to include an overall factor equal to $\mu^{2\epsilon}$ to keep the coupling constant dimensionless and a factor of $(2\pi)^{2\epsilon}$ due to phase space. There is also the factor $(1-\epsilon)$ that comes from the Dirac algebra in n-dimensions.

Putting this together with equations (4.82) and (B.7) we find the Fourier transform of the divergent part of the real contribution of the cross section;

$$
\begin{aligned}
&\int d^n\vec{Q}_T e^{-i\vec{Q}_T\cdot\vec{b}}\frac{d\sigma^{(R)}}{dQ^2dyd^2\vec{Q}_T} \\
&\quad\approx N\frac{(1-\epsilon)}{2\pi}\left(\frac{\alpha_S(Q)}{\pi}\right)(2\pi\mu)^{2\epsilon}\left\{I_0\left[\delta(1-z_B)P^{(1)}_{j/k}(z_A)\right.\right. \\
&\quad + \delta(1-z_A)P^{(1)}_{\bar{j}/k}(z_B)+2\delta(1-z_A)\delta(1-z_B)\left(\frac{4}{3}\ln Q^2-2\right) \\
&\quad \left.-\epsilon\frac{4}{3}\left(\delta(1-z_A)(1-z_B)+\delta(1-z_B)(1-z_A)\right)\right] \\
&\quad \left.-2\frac{4}{3}\delta(1-z_A)\delta(1-z_B)I_1\right\} \qquad (4.92)
\end{aligned}
$$

where $N\equiv\sum_j e_j^2\frac{4\pi\alpha^2}{9Q^2S}$ and (see Appendix C)

$$I_0 \equiv \int\frac{d^n\vec{Q}_T}{Q_T^2}e^{-i\vec{Q}_T\cdot\vec{b}}=\left(\frac{2}{b}\right)^{n-2}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}-1\right) \qquad (4.93)$$

$$I_1 \equiv \int\frac{d^n\vec{Q}_T}{Q_T^2}e^{-i\vec{Q}_T\cdot\vec{b}}\ln Q_T^2=\left(\frac{2}{b}\right)^{n-2}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}-1\right)\left[\psi\left(\frac{n}{2}-1\right)-\gamma_E-\ln\left(\frac{b^2}{4}\right)\right] \qquad (4.94)$$

with $\Gamma(z)$ the gamma or factorial function, $\psi(z)$ is the logarithmic derivative of the gamma function and γ_E is the Euler's constant. Taking $n=2-2\epsilon$ we find:

$$I_0 = -\pi\left(\frac{b^2}{4\pi}\right)^{\epsilon}\left[\frac{1}{\epsilon}+\gamma_E+\mathcal{O}(\epsilon)\right] \qquad (4.95)$$

$$I_1 = = -\pi^{1-\epsilon}\left[\frac{1}{\epsilon^2}-\frac{\gamma_E}{\epsilon}-\frac{\pi^2}{12}-\frac{3}{2}\gamma_E^2-2\gamma_E\ln\frac{b^2}{4}-\frac{1}{2}\ln^2\frac{b^2}{4}+\mathcal{O}(\epsilon)\right] \qquad (4.96)$$

which allow us to calculate

$$(2\pi\mu)^{2\epsilon}I_0 = -\pi\left(\frac{1}{\epsilon}+\gamma_E+\ln\left(b^2\pi\mu^2\right)+\mathcal{O}(\epsilon)\right) \qquad (4.97)$$

$$
\begin{aligned}
(2\pi\mu)^{2\epsilon}I_1 &= -\pi\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\ln(4\pi\mu^2)-\gamma_E\right)-2\gamma_E\ln\left(\frac{b^2}{4}\right)-\frac{3}{2}\gamma_E^2-\frac{\pi^2}{12}\right. \\
&\left.-\frac{1}{2}\ln^2\left(\frac{b^2}{4}\right)+\frac{1}{2}\ln^2(4\pi\mu^2)-\gamma_E\ln(4\pi\mu^2)+\mathcal{O}(\epsilon)\right] \qquad (4.98)
\end{aligned}
$$

Now the virtual part. The evaluation of the Fourier transform is trivial due to the presence of the $\delta^2(\vec{Q}_T)$ in the phase space thus:

$$\int d^2\vec{Q}_T e^{-i\vec{Q}_T\cdot\vec{b}} \frac{d\sigma^{(V)}}{dQ^2dyd^2\vec{Q}_T}$$
$$= N(1-\epsilon)\frac{-4}{3}\left(\frac{\alpha_S(Q)}{\pi}\right)\left[\frac{4\pi\mu^2}{-Q^2}\right]^\epsilon \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}+4\right)\delta(1-z_A)\delta(1-z_B)$$
$$= N(1-\epsilon)\frac{-4}{3}\left(\frac{\alpha_S(Q)}{\pi}\right)\left\{\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left[\frac{3}{2}-\gamma_E+\ln\left(\frac{4\pi\mu^2}{Q^2}\right)\right]+4-\frac{3}{2}\gamma_E+\frac{1}{2}\gamma_E^2-\frac{7}{12}\pi^2\right.$$
$$\left.+\ln\left(\frac{4\pi\mu^2}{Q^2}\right)\left(\frac{3}{2}-\gamma_E\right)+\frac{1}{2}\ln^2\left(\frac{4\pi\mu^2}{Q^2}\right)+\mathcal{O}(\epsilon)\right\} \tag{4.99}$$

Here equations (B.9), (2.24), (3.3) and (3.12) were used. We can observe here that the real and virtual parts contain divergent parts proportional to $1/\epsilon^2$ and $1/\epsilon$. The first term comes from the superposition of soft and collinear singularities. In the virtual diagram the $1/\epsilon$ term comes from either soft or collinear singularities, while in the real contribution only collinear singularities are present.

Adding equations (4.92) and (4.99) we get:

$$\int d^{2-2\epsilon}\vec{Q}_T e^{-i\vec{Q}_T\cdot\vec{b}}\left(\frac{d\sigma^{(R)}}{dQ^2dyd^2\vec{Q}_T}+\frac{d\sigma^{(V)}}{dQ^2dyd^2\vec{Q}_T}\right)$$
$$= N(1-\epsilon)\left(\frac{\alpha_S(Q)}{\pi}\right)\left\{-\frac{1}{2\epsilon}\left[\frac{4}{3}\left(\frac{1+z_A^2}{1-z_A}\right)_+\delta(1-z_B)+\frac{4}{3}\left(\frac{1+z_B^2}{1-z_B}\right)_+\delta(1-z_A)\right]\right.$$
$$+\delta(1-z_A)\delta(1-z_B)\left[\frac{2}{3}\pi^2-\frac{23}{6}-\frac{8}{3}\ln^2\left(\frac{Qb\,e^{\gamma_E-3/4}}{2}\right)\right]$$
$$-\left[\frac{2}{3}\left(\frac{1+z_A^2}{1-z_A}\right)_+\delta(1-z_B)+\frac{2}{3}\left(\frac{1+z_B^2}{1-z_B}\right)_+\delta(1-z_A)\right]\left(\gamma_E+\ln(b^2\pi\mu^2)\right)$$
$$\left.+\frac{2}{3}\left[\delta(1-z_A)(1-z_B)+\delta(1-z_B)(1-z_A)\right]\right\} \tag{4.100}$$

The first remarkable thing to observe is the cancellation of the terms proportional to $1/\epsilon^2$. This is an example of the Kinoshita-Lee-Navenberg theorem [98] where the soft divergences of the virtual gluon correction to $q\bar{q}$ cancel with the soft divergences of the $q\bar{q}G$ final state. Now the term proportional to $1/\epsilon$ is absorbed by the distribution functions by the Factorization Theorem in the Drell-Yan process Eq.(B.6). So we are left with a collection of finite terms.

These numbers are equal to:

$$
\begin{aligned}
&C_{ja}^{(1)}\left(z_A, b; \frac{c_1}{c_2}; Q\right)\delta(1-z_B) + C_{\bar{j}b}^{(1)}\left(z_B, b; \frac{c_1}{c_2}; Q\right)\delta(1-z_A) \\
&\quad = \delta(1-z_A)\delta(1-z_B)\left[\frac{2}{3}\pi^2 - \frac{23}{6} - \frac{8}{3}\ln^2\left(\frac{c_1\, e^{\gamma_E - 3/4}}{2c_2}\right)\right] \\
&\quad\quad - \left[\frac{2}{3}\left(\frac{1+z_A^2}{1-z_A}\right)_+ \delta(1-z_B) + \frac{2}{3}\left(\frac{1+z_B^2}{1-z_B}\right)_+ \delta(1-z_A)\right]\ln\left(\frac{e^{2\gamma_E} b^2 \mu_{\overline{\mathrm{MS}}}^2}{4}\right) \\
&\quad\quad + \frac{2}{3}\left[\delta(1-z_A)(1-z_B) + \delta(1-z_B)(1-z_A)\right] \qquad (4.101)
\end{aligned}
$$

Here equations (4.38), (4.67), (4.68) and (4.90) where used together with $\mu_{\overline{\mathrm{MS}}} = \mu_{MS}\, e^{(\ln(4\pi)-\gamma_E)/2}$ at NLO [91]. Thus we can conclude [13], [44]:

$$
\begin{aligned}
&C_{ja}^{(1)}\left(z, b; \frac{c_1}{c_2}; Q\right) \\
&\quad = \delta_{ja}\left\{\delta(1-z)\left[\frac{1}{3}\pi^2 - \frac{23}{12} - \frac{4}{3}\ln^2\left(\frac{c_1\, e^{\gamma_E - 3/4}}{2c_2}\right)\right] - \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)_+ \ln\left(\frac{e^{\gamma_E} b \mu_{\overline{\mathrm{MS}}}}{2}\right) + \frac{2}{3}(1-z)\right\}
\end{aligned} \qquad (4.102)
$$

To calculate the C-function for the Compton subprocess we proceed likewise. But we need to include the factor $2(1-\epsilon)$ to account for the degrees of freedom of the gluon in $n = 2(1-\epsilon)$ dimensions. From equations (B.10) and (4.7) we find:

$$
\begin{aligned}
&\int d^{2-2\epsilon}\vec{Q}_T e^{-i\vec{Q}_T\cdot\vec{b}} \frac{d\sigma^{(qG)}}{dQ^2 dy d^2\vec{Q}_T} \\
&= N\left(\frac{\alpha_S(Q)}{\pi}\right)\frac{(2\pi\mu)^{2\epsilon}}{2(1-\epsilon)\pi} I_0 \left\{\frac{1}{2}\left[(1-z_B)^2 + z_B^2\right](1-\epsilon) + \epsilon z_B(z_B - 1)\right\}\delta(1-z_A) \\
&= N\left(\frac{\alpha_S(Q)}{\pi}\right)\left\{-\frac{1}{2\epsilon}\frac{1}{2}\left[(1-z_B)^2 + z_B^2\right] + \frac{z_B(1-z_B)}{2}\right. \\
&\qquad \left. - \frac{1}{2}\ln\left(\frac{e^{\gamma_E} b\mu_{\overline{\mathrm{MS}}}}{2}\right)\left[(1-z_B)^2 + z_B^2\right]\right\}\delta(1-z_A) + \mathcal{O}(\epsilon) \qquad (4.103)
\end{aligned}
$$

Again the divergent part is absorbed by the distribution functions. Therefore we deduce that [44], [13]:

$$
C_{jg}^{(1)}\left(z, b; \frac{c_1}{c_2}; Q\right) = \frac{z_B(1-z_B)}{2} - \frac{1}{2}\ln\left(\frac{e^{\gamma_E} b\mu_{\overline{\mathrm{MS}}}}{2}\right)\left[(1-z_B)^2 + z_B^2\right] \qquad (4.104)
$$

With the canonical choices for $\mu = \mu_{\overline{\mathrm{MS}}}$, c_1 and c_2 expressions (4.102) and (4.104) are simplified to:

$$C_{ja}^{(1)}(z) = \delta_{ja}\left[\delta(1-z)\left(\frac{1}{3}\pi^2 - \frac{8}{3}\right) + \frac{2}{3}(1-z)\right] \tag{4.105}$$

$$C_{jg}^{(1)}(z) = \frac{z(1-z)}{2} \tag{4.106}$$

which can also be found in [44] or [63].

4.3.5 Finite Part

To finish our description of the resummed formula Eq.(4.20), we need to exhibit formulas for the Y_f part defined in Eq.(4.21). The Y_f term is equal to the difference of the fixed order perturbative result and their low Q_T limit. Let us start with the $q\bar{q}$ process, we have at NLO:

$$\begin{aligned} R^1_{j\bar{j}} &= R^1_{\bar{j}j} \\ &= \frac{2}{3\pi Q_T^2}\left\{\frac{(Q^2-t)^2+(Q^2-u)^2}{s}\delta(s+t+u-Q^2)\right. \\ &\left. -\delta(1-z_B)\left[\frac{1+z_A^2}{1-z_A}\right]_+ - \delta(1-z_A)\left[\frac{1+z_B^2}{1-z_B}\right]_+ - 2\delta(1-z_A)\delta(1-z_B)\left(\ln Q^2 - \frac{3}{2}\right)\right\} \end{aligned} \tag{4.107}$$

where equations (3.105) and (4.82) have been used. For the Compton contribution, qg subprocess we obtain using equations (3.106) and (4.103)

$$\begin{aligned} \mathcal{R}^1_{jg} &= \mathcal{R}^1_{\bar{j}g} \\ &= \frac{1}{4\pi}\left\{\frac{(Q^2-s)^2+(Q^2-u)^2}{-us}\delta(s+t+u-Q^2) - \frac{1}{Q_T^2}\left[(1-z_B)^2+z_B^2\right]\delta(1-z_A)\right\} \end{aligned} \tag{4.108}$$

and similarly for the gq subprocess

$$\begin{aligned} \mathcal{R}^1_{gj} &= \mathcal{R}^1_{g\bar{j}} \\ &= \frac{1}{4\pi}\left\{\frac{(Q^2-s)^2+(Q^2-t)^2}{-ts}\delta(s+t+u-Q^2) - \frac{1}{Q_T^2}\left[(1-z_A)^2+z_A^2\right]\delta(1-z_B)\right\} \end{aligned} \tag{4.109}$$

At this order any other possible contributions like $\mathcal{R}^1_{gg}$, $\mathcal{R}^1_{jj'}$, $\mathcal{R}^1_{\bar{j}j'}$ do not contribute [44].

CHAPTER 5. RESUMMATION AND STRUCTURE FUNCTIONS

In this chapter we will apply resummation to the fully differential cross section. In the first section we apply resummation only to the transverse structure function. Then we explore how to extend resummation to the remaining functions. Conclusions are also included.

5.1 First extension

The most direct way to extend resummation to the fully differential cross section of the Drell-Yan process is to apply this technique to each one of the structure functions defined in Chapter 2. Unfortunately, only W_T has the right structure. Let us understand this better. By Eq.(2.24) the integrated cross section is proportional to the sum $2W_T + W_L$, but the transverse structure function dominates at low Q_T, thus:

$$\lim_{Q_T \to 0} \frac{d\sigma}{d^4q} = \lim_{Q_T \to 0} \frac{\alpha^2}{6S^2Q^2\pi^3} W_T \tag{5.1}$$

since the singularity for W_T is the dominant singularity[1]. Hence, when we performed the resummation of the integrated cross section we were actually resumming W_T. Clearly this implies for the differential cross section:

$$\lim_{Q_T \to 0} \frac{d\sigma}{d^4q d\Omega} = \lim_{Q_T \to 0} \frac{\alpha^2}{2S^2Q^2\left(2\pi\right)^4} \left[W_T\left(1+\cos^2\theta\right)\right] \tag{5.2}$$

and therefore at low Q_T we have:

$$\frac{d\sigma}{d^4q d\Omega} \approx \frac{\alpha^2}{2S^2Q^2\left(2\pi\right)^4} \left[W_T^{Resum.}\left(1+\cos^2\theta\right)\right] \tag{5.3}$$

[1]The fact that this singularity is proportional to $\frac{1}{Q_T^2}$ is fundamental for resummation

and for $0 < Q_T \approx Q$

$$
\begin{aligned}
&\frac{d\sigma}{d^4q d\Omega} \\
&= \frac{\alpha^2}{2S^2Q^2(2\pi)^4}\left[W_T^{Resum.}\left(1+\cos^2\theta\right)\right] \\
&+\frac{\alpha^2}{2S^2Q^2(2\pi)^4}\left[W_T^f\left(1+\cos^2\theta\right)+W_L^f\left(1-\cos^2\theta\right)+W_{\Delta\Delta}^f\cos 2\phi\sin^2\theta+W_\Delta^f\sin 2\theta\cos\phi\right]
\end{aligned}
\tag{5.4}
$$

with [12],[13] and [63]:

$$
\begin{aligned}
&W_T^{Resum.} \\
&= \frac{(2\pi)^4 S}{3}\frac{1}{(2\pi)^2}\int d^2b e^{i\vec{Q_T}\cdot\vec{b}}\sum_j e_j^2\sum_{ab}F_{ab}^{NP}\left(Q,b,x_A,x_B\right) \\
&\times\int_{x_A}^1\frac{d\xi_A}{\xi_A}f_{a/A}\left(\xi_A;1/b_*\right)\int_{x_B}^1\frac{d\xi_B}{\xi_B}f_{b/B}\left(\xi_B;1/b_*\right) \\
&\times\exp\left\{-\int_{1/b_*^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\ln\left(\frac{Q^2}{\bar{\mu}^2}\right)A(g(\bar{\mu}))+B(g(\bar{\mu}))\right]\right\} \\
&\times C_{ja}\left(\frac{x_A}{\xi_A};g(1/b_*)\right)C_{\bar{j}a}\left(\frac{x_B}{\xi_B};g(1/b_*)\right)
\end{aligned}
\tag{5.5}
$$

where F_{ab}^{NP} and b_* have been explained at the beginning of section 4.3.2 and the functions A, B and C are the same functions found in the last chapter. The finite pieces are:

For W_T

$q\bar{q}$ subprocess:

$$
\begin{aligned}
&W_{T(q\bar{q})}^{finite} = W_{T(q\bar{q})}^{pert} - W_{T(q\bar{q})}^{asymp} \\
&= \frac{(2\pi)^4 S}{3}\left(\frac{\alpha_S(Q)}{\pi}\right)\frac{2}{3\pi}\left\{\frac{1}{2}\left(1+\frac{2Q^2}{Q_T^2}\right)\left(\frac{z_A}{z_B}+\frac{z_B}{z_A}\right)\delta(s+t+u-Q^2)\right. \\
&\left.-\frac{1}{Q_T^2}\left[\delta(1-z_B)\left[\frac{1+z_A^2}{1-z_A}\right]_+ +\delta(1-z_A)\left[\frac{1+z_B^2}{1-z_B}\right]_+ +2\delta(1-z_A)\delta(1-z_B)\left(\ln\frac{Q^2}{Q_T^2}-\frac{3}{2}\right)\right]\right\}
\end{aligned}
\tag{5.6}
$$

qg subprocess:

$$\begin{aligned}
W^{finite}_{T(qg)} &= W^{pert}_{T(qg)} - W^{asymp}_{T(qg)} \\
&= \frac{(2\pi)^4 S}{3}\left(\frac{\alpha_S(Q)}{\pi}\right)\frac{1}{4\pi Q_T^2}\{ \\
& -\frac{1}{2}\left[(z_A - z_B)^2 + z_B^2 + (1 - z_A z_B)^2 + z_B^2 + z_B^2\frac{Q_T^2}{Q^2}\right]\frac{Q^2}{z_B}\left(\sqrt{1+\frac{Q_T^2}{Q^2}} - z_B\right)\delta(s+t+u-Q^2) \\
& - \left[(1-z_B)^2 + z_B^2\right]\delta(1-z_A)\}
\end{aligned} \tag{5.7}$$

for gq subprocess:

$$\begin{aligned}
W^{finite}_{T(gq)} &= W^{pert}_{T(gq)} - W^{asymp}_{T(gq)} \\
&= \frac{(2\pi)^4 S}{3}\left(\frac{\alpha_S(Q)}{\pi}\right)\frac{1}{4\pi Q_T^2}\{ \\
& -\frac{1}{2}\left[(z_B - z_A)^2 + z_A^2 + (1 - z_B z_A)^2 + z_A^2 + z_A^2\frac{Q_T^2}{Q^2}\right]\frac{Q^2}{z_A}\left(\sqrt{1+\frac{Q_T^2}{Q^2}} - z_A\right)\delta(s+t+u-Q^2) \\
& - \left[(1-z_A)^2 + z_A^2\right]\delta(1-z_B)\}
\end{aligned} \tag{5.8}$$

For the finite part of the other structure functions we just have to recall the expressions for W_L, W_Δ and $W_{\Delta\Delta}$ present in Chapter 4 and in Tables 3.3, 3.5 and 3.7.

We define here:

$$W_T^{Total} = W_T^{resummed} + W_T^{finite} \tag{5.9}$$

and observe in Fig. 5.1 the prediction for W_T^{Total} vs Q_T. Based on this graph and comparing with Figures 3.14-3.17 we can conclude that:

$$\lambda \approx 1 \quad \nu \approx 0 \quad \mu \approx 0 \tag{5.10}$$

which can also be seen in Figures 5.2, 5.3, and 5.4 for $p\bar{p}$ in the Collins-Soper frame with $Q = 10\ GeV/c$, $y = 0$ and $\sqrt{S} = 800\ GeV/c$. For the nonperturbative part, the BLNY parameterization was used [89]:

$$F^{NP}\left(Q, b, b_{max}, x_A, x_B\right) = \exp\left[-b^2\left(g_1 + g_1 g_3 \ln(100\, x_A\, x_B)) + g_2 \ln\frac{Q}{2Q_0}\right)\right] \tag{5.11}$$

with the following values:

$$
\begin{aligned}
g_1 &= 0.21\ GeV^2 \\
g_2 &= 0.68\ GeV^2 \\
g_3 &= -0.6 \\
Q_0 &= 1.6\ GeV \\
b_{max} &= 0.5\ GeV^{-1}\quad for\ b_* \qquad (5.12)
\end{aligned}
$$

where b_* was defined in Eq.(4.55)

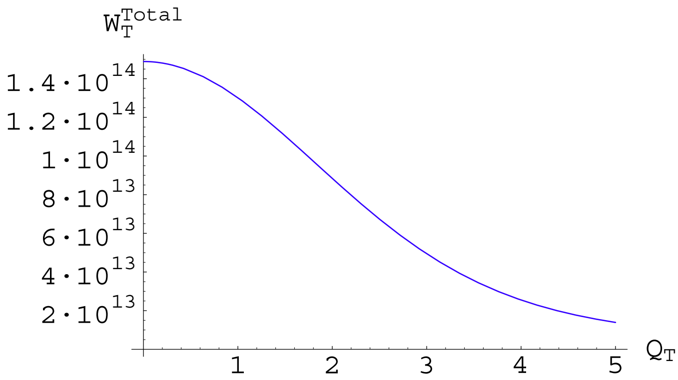


Figure 5.1 $W_T{}^{Total}$ vs Q_T

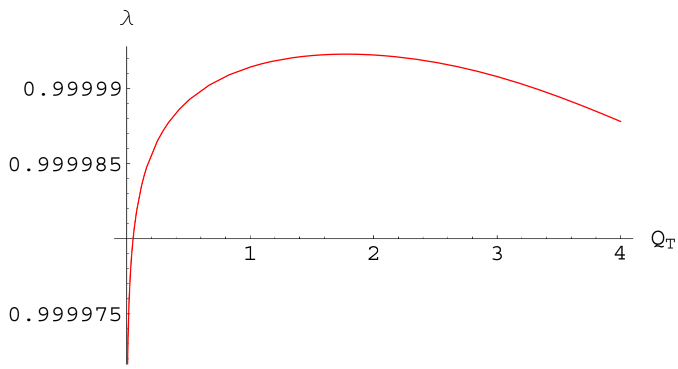


Figure 5.2 λ vs Q_T resummation prediction

We can understand the above results as consequence of the hierarchy of divergences for each of the structure functions and the corresponding resummation performed on W_T. Since

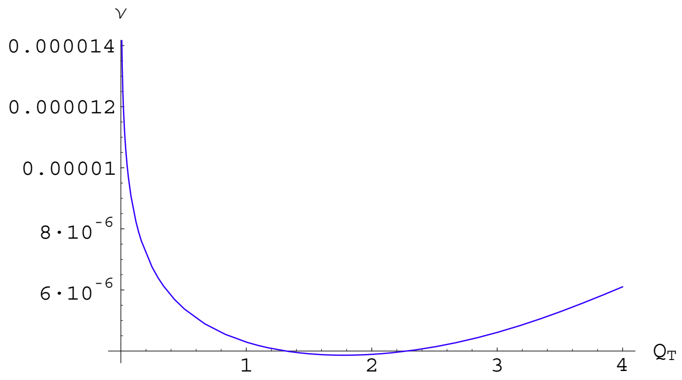


Figure 5.3 ν vs Q_T resummation prediction

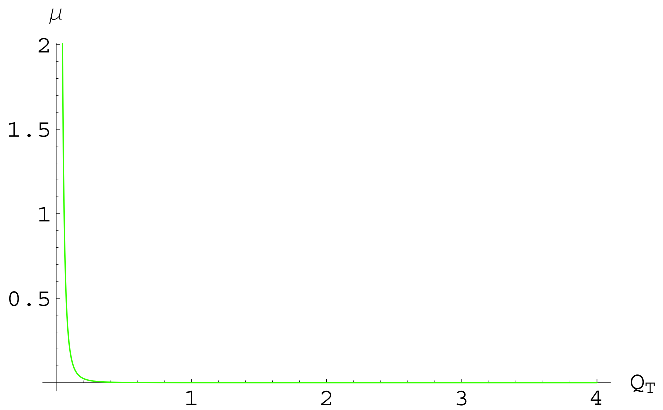


Figure 5.4 μ vs Q_T resummation prediction

$1/Q_T^2$ is the dominant divergence and W_L and $W_{\Delta\Delta}$ are finite at low Q_T, we have that W_T^{pert} is the dominant structure function. This also implies that $W_T^{resummed}$ is the dominant quantity, except for a region of very low Q_T, where the divergence of W_Δ dominates. We will ignore this region, since we cannot trust perturbative results there.

Now, let us remember the definitions of λ, ν and μ Eq.(2.26):

$$\lambda = \frac{W_T - W_L}{W_T + W_L} \quad \mu = \frac{W_\Delta}{W_T + W_L} \quad \nu = \frac{2W_{\Delta\Delta}}{W_T + W_L}$$

thus, including the results from resummation we have:

$$\lambda = \frac{W_T^{Total} - W_L^{finite}}{W_T^{Total} + W_L^{finite}} \quad \mu = \frac{W_\Delta^{finite}}{W_T^{Total} + W_L^{finite}} \quad \nu = \frac{2W_{\Delta\Delta}^{finite}}{W_T^{Total} + W_L^{finite}}$$

thus, at low Q_T but different from 0 they become:

$$\lambda = \frac{W_T^{Total}}{W_T^{Total} + W_L^{finite}} \approx 1 \quad \mu = \frac{1}{W_T^{Total}} \approx 0 \quad \nu = \frac{1}{W_T^{Total}} \approx 0$$

From these outcomes we can deduce the validity of the Lam-Tum relation for the resummed predictions at low Q_T. Observe that the above predictions are compatible with the experimental results from E866 (See Fig. 1.2).

The validity of the Lam-Tung is also implied by

$$\frac{d\sigma}{d^4q} \approx \frac{\alpha^2}{12S^2Q^2\pi^3}\left(2W_T^{Total}\right)$$

which follows the by discussion that appears after Eq.(3.25).

Conceptually, the above extension is not satisfactory since we are mixing fixed order perturbative predictions with resummed results. We would like to apply resummation to the other structure functions and check if we can obtain a violation of the Lam-Tum relation and a ν behavior closed to the measured one. We also want a method of resummation that can be independent of the photon rest frame used.

5.2 Second extension

Let us begin this section with the following claim:

Divergent part form. *The divergent part at low Q_T of the Drell-Yan parton tensor has the following form at any order in the perturbative expansion:*

$$\begin{aligned} w^{\mu\nu\ diverg} &\equiv \lim_{Q_T \to 0} w^{\mu\nu} \\ &= \mathcal{F}_0\, d^{\mu\nu} + \mathcal{F}_1 \frac{Q_T}{Q_-}\left(n_+^\mu n_T^\nu + n_+^\nu n_T^\mu\right) + \mathcal{F}_2 \frac{Q_T}{Q_+}\left(n_-^\mu n_T^\nu + n_-^\nu n_T^\mu\right) + \mathcal{F}_3\left(n_T^\mu n_T^\nu\right) + \mathcal{F}_4\, d^{\mu\nu} \end{aligned} \tag{5.13}$$

where[2]*:*

$$d^{\mu\nu} = n_+^\mu n_-^\nu + n_+^\nu n_-^\mu - g^{\mu\nu}$$

[2] *Vectors n^+,n^- and n^T were defined in Eq.(3.34)*

and the $\mathcal{F}_i$ are scalar functions of the relevant kinematic variables. These functions are such that $\mathcal{F}_1 + \mathcal{F}_2 = \mathcal{F}_0$ with the following behavior at low Q_T: $\mathcal{F}_0 \propto 1/Q_T^2$ and for the rest of them either there are divergent or zero. If they are divergent then $\mathcal{F}_1, \mathcal{F}_2 \propto 1/Q_T^2$ and $\mathcal{F}_3, \mathcal{F}_4 \propto 1/Q_T$

Proof. Let us start with the following basic observation. at $Q_T \approx 0$ we have only two vectors: p_A, p_B from which we can only form one symmetric tensor:

$$\mathcal{D}_1 \left(p_A^\mu p_B^\nu + p_A^\nu p_B^\mu\right) + \mathcal{D}_2 \, g^{\mu\nu}$$

with $\mathcal{D}_1$ and $\mathcal{D}_2$ two arbitrary scalar functions. Now, if we require gauge invariance these two functions are not longer independent. Then, we get [3]:

$$\mathcal{F} \left(p_A^\mu p_B^\nu + p_A^\nu p_B^\mu - \frac{s}{2}\, g^{\mu\nu}\right)$$

It is important to notice that this tensor is gauge invariant only when $Q_T = 0$. Using definitions (3.34), and equations (3.42) and (3.43), together with (3.40) and (3.41), we can write this tensor as $\mathcal{F}_0 \, d^{\mu\nu}$. The fact that $d^{\mu\nu}$ is unique in the low Q_T limit allows us to conclude the following relation between this tensor and the partonic tensor for the Drell-Yan process:

$$w^{\mu\nu \; diverg} = \left(\lim_{Q_T \to 0} \mathcal{F}_0(Q_T)\right) d^{\mu\nu} \tag{5.14}$$

Notice also that $d^{\mu\nu}$ only has a nonzero projection along the transverse structure function[4]. Thus, we can identify $\mathcal{F}_0$ with w_T at any order in perturbation theory. The general form of W_T is known:

$$W_T^n \propto T_{ab}^n \left(Q_T, Q, \frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}; \mu\right) \tag{5.15}$$

with T_{ab}^n defined in Eq.(4.19). Then

$$\lim_{Q_T \to 0} \mathcal{F}_0^n(Q_T) = \frac{1}{Q_T^2} \ln^{2n-1}\left(\frac{Q^2}{Q_T^2}\right) \tag{5.16}$$

taking into account only the $1/Q_T^2$ divergence we observe for low Q_T but different from zero at any order in the perturbative expansion:

$$\mathcal{F}_0(Q_T) d^{\mu\nu} q_\mu = -Q_T \, \mathcal{F}_0(Q_T) n_T^\nu \tag{5.17}$$

[3] Compare with Eq.(3.17), where $\mathcal{F} = 1$.
[4] This can seen from Table 3.1 and Eq.(3.17)

The scalar function in the term at the right hand of Eq.(5.17) is proportional to $1/Q_T$ and accompanies the vector n_T^ν. In order to preserve gauge invariance, we need to consider the following tensors[5]: $n_+^\mu n_T^\nu + n_+^\nu n_T^\mu$, $n_-^\mu n_T^\nu + n_-^\nu n_T^\mu$, $n_T^\nu n_T^\mu$ and $d^{\mu\nu}$ and the corresponding scalar functions[6]. These are the only tensors that contracted with q^μ yield a term proportional to n_T^ν. Notice that the function accompanying $d^{\mu\nu}$ can only be at most proportional to $1/Q_T$, otherwise it will be included in $\mathcal{F}_0$. A companion of $n_T^\nu n_T^\mu$ proportional to $1/Q_T^2$ contradicts the fact that w_T is the only structure function with singularities proportional to $1/Q_T^2$. We define $\mathcal{F}_1$ and $\mathcal{F}_2$ as the scalar functions of $\frac{Q_T}{Q_-}\left(n_+^\mu n_T^\nu + n_+^\nu n_T^\mu\right)$ and $\frac{Q_T}{Q_+}\left(n_-^\mu n_T^\nu + n_-^\nu n_T^\mu\right)$ respectively with

$$Q_+ = \frac{Q\, e^{y}}{\sqrt{2}}\sqrt{1+\frac{Q_T^2}{Q^2}} \tag{5.18}$$

$$Q_- = \frac{Q\, e^{-y}}{\sqrt{2}}\sqrt{1+\frac{Q_T^2}{Q^2}} \tag{5.19}$$

In similar way we define $\mathcal{F}_3$ and $\mathcal{F}_4$. The relation $\mathcal{F}_1 + \mathcal{F}_2 = \mathcal{F}_0$ is consequence of gauge invariance[7]. $\square$

Despite the fact that we have used gauge invariance to find the general form of Eq.(5.13), this tensor is not gauge invariant in general. Therefore, we need to find a tensor that conserves current and includes as much as we can of tensor (5.13). Inspired by the form of the tensors for the annihilation and Compton subprocesses, we construct order by order in perturbation theory the following gauge invariant combination:

$$k^{\mu\nu} \equiv \mathcal{F}_0\, d^{\mu\nu} + \mathcal{F}_1 \frac{Q_T}{Q_-}\left(n_+^\mu n_T^\nu + n_+^\nu n_T^\mu + \frac{Q_T}{Q_-} n_+^\mu n_+^\nu\right) + \mathcal{F}_2 \frac{Q_T}{Q_+}\left(n_-^\mu n_T^\nu + n_-^\nu n_T^\mu + \frac{Q_T}{Q_+} n_-^\mu n_-^\nu\right) \tag{5.20}$$

We remark here that this tensor contains the most divergent part of $w^{\mu\nu}$ plus a certain finite

[5]Remember that $w^{\mu\nu}$ is symmetric.

[6]We could have started with the general set $n_+^\mu n_T^\nu + n_+^\nu n_T^\mu$,$n_-^\mu n_T^\nu + n_-^\nu n_T^\mu$,$n_T^\nu n_T^\mu$, $g^{\mu\nu}$ and $n_+^\mu n_-^\nu + n_+^\nu n_-^\mu$ and reduce its independent elements using gauge invariance and the behavior of the scalar functions at low Q_T; either way you could have ended with the same set of tensors and scalar functions.

[7]We proved in Sec.(2.3) that there are only 4 independent structure functions at any given transverse momentum for $W^{\mu\nu}$.

part necessary to assure $k^{\mu\nu} q_\mu = 0$. We define a new finite tensor by

$$f^{\mu\nu} \equiv w^{\mu\nu} - k^{\mu\nu} \tag{5.21}$$

notice that $f^{\mu\nu}$ is gauge invariant and finite or at most with a divergence proportional to $1/Q_T$ when $Q_T \to 0$.

We can observe this kind of decomposition applied to the annihilation subprocess:

$$\begin{aligned} & w_{q\bar{q}}^{\mu\nu\,(NLO)} \\ = \; & \frac{1}{sQ_T^2}\left[(Q^2-t)^2+(Q^2-u)^2\right]\delta\left(s+t+u-Q^2\right)d^{\mu\nu} \\ & + \left[\frac{(Q^2-u)^2}{sQ_T^2}\delta\left(s+t+u-Q^2\right)\right]\frac{Q_T}{Q_-}\left(n_+^\mu n_T^\nu + n_+^\nu n_T^\mu + \frac{Q_T}{Q_-} n_+^\mu n_+^\nu\right) \\ & + \left[\frac{(Q^2-t)^2}{sQ_T^2}\delta\left(s+t+u-Q^2\right)\right]\frac{Q_T}{Q_+}\left(n_-^\mu n_T^\nu + n_-^\nu n_T^\mu + \frac{Q_T}{Q_+} n_-^\mu n_-^\nu\right) \end{aligned}$$

with $\mathcal{F}_1 = \frac{(Q^2-u)^2}{sQ_T^2}\delta\left(s+t+u-Q^2\right)$, $\mathcal{F}_2 = \frac{(Q^2-t)^2}{sQ_T^2}\delta\left(s+t+u-Q^2\right)$ and $f^{\mu\nu} = 0$. For the Compton subprocess we have:

$$\begin{aligned} & w_{qg}^{\mu\nu\,(NLO)} \\ = \; & \frac{1}{-2su}\left(u^2+s^2+2tQ\right)\delta\left(s+t+u-Q^2\right)d^{\mu\nu} \\ & + \left[\frac{(Q^2+t+2s)(t+s)-2sQ^2}{-2su}\delta\left(s+t+u-Q^2\right)\right]\frac{Q_T}{Q_-}\left(n_+^\mu n_T^\nu + n_+^\nu n_T^\mu + \frac{Q_T}{Q_-} n_+^\mu n_+^\nu\right) \\ & + \left[\frac{(Q^2+s)(Q^2-t)-2sQ^2}{-2su}\delta\left(s+t+u-Q^2\right)\right]\frac{Q_T}{Q_+}\left(n_-^\mu n_T^\nu + n_-^\nu n_T^\mu + \frac{Q_T}{Q_+} n_-^\mu n_-^\nu\right) \end{aligned}$$

with

$$\begin{aligned} \mathcal{F}_1 &= \frac{(Q^2+t+2s)(t+s)-2sQ^2}{-2su}\delta\left(s+t+u-Q^2\right) \\ \mathcal{F}_2 &= \frac{(Q^2+s)(Q^2-t)-2sQ^2}{-2su}\delta\left(s+t+u-Q^2\right) \end{aligned}$$

and

$$
\begin{aligned}
& f_{qg}^{\mu\nu\,(NLO)} \\
& = \quad \Big\{ Q_T^2 s \left[\left(n_+^\mu n_-^\nu + n_+^\nu n_-^\mu\right) - \frac{Q^2}{2Q_-^2} n_+^\mu n_+^\nu - \frac{Q^2}{2Q_+^2} n_-^\mu n_-^\nu + 2 n_T^\nu n_T^\mu \right] \\
& \quad + Q_T^3 s \left[\frac{1}{Q_-} \left(n_+^\mu n_T^\nu + n_+^\nu n_T^\mu\right) + \frac{1}{Q_+} \left(n_-^\mu n_T^\nu + n_-^\nu n_T^\mu\right) \right] \\
& \quad + Q_T^4 s \left[\frac{1}{2Q_+^2} n_-^\mu n_-^\nu + \frac{1}{2Q_-^2} n_+^\mu n_+^\nu \right] \Big\} \, \delta\left(s + t + u - Q^2\right)
\end{aligned}
$$

The existence of $k^{\mu\nu}$ allows us to find the following relations:

$$
w_L = k^{\mu\nu} \hat{z}_\mu \hat{z}_\nu = \frac{Q_T^2}{Q^2 + Q_T^2} \left(\mathcal{F}_1 + \mathcal{F}_2\right) \tag{5.22}
$$

where we have used

$$
\hat{z}_\mu n_+^\mu = -\frac{e^{-y}}{\sqrt{2}}, \quad \hat{z}_\mu n_-^\mu = \frac{e^{y}}{\sqrt{2}}, \quad \hat{z}_\mu n_T^\mu = 0 \tag{5.23}
$$

and

$$
w_T = \left[1 - \frac{Q_T^2}{2\left(Q^2 + Q_T^2\right)} \right] \left(\mathcal{F}_1 + \mathcal{F}_2\right) \tag{5.24}
$$

here we have employed the projector operators defined in Eq.(2.21).

In like manner:

$$
w_{\Delta\Delta} = w_T - k^{\mu\nu} \hat{x}_\mu \hat{x}_\nu = \frac{1}{2} \frac{Q_T^2}{Q^2 + Q_T^2} \left(\mathcal{F}_1 + \mathcal{F}_2\right) \tag{5.25}
$$

where we have drawn upon:

$$
\hat{x}_\mu n_+^\mu = \frac{e^{-y}}{\sqrt{2}} \frac{Q_T}{Q}, \quad \hat{x}_\mu n_-^\mu = \frac{e^{y}}{\sqrt{2}} \frac{Q_T}{Q}, \quad \hat{x}_\mu n_T^\mu = -\sqrt{1 + \frac{Q_T^2}{Q^2}} \tag{5.26}
$$

and finally,

$$
w_\Delta = -k^{\mu\nu} \hat{z}_\mu \hat{x}_\nu = \frac{Q_T}{Q} \left(1 - \frac{Q_T^2}{Q^2 + Q_T^2} \right) \left(\mathcal{F}_2 - \mathcal{F}_1\right) \tag{5.27}
$$

The above relations are valid at any order in the perturbative expansion, thus:

$$
\begin{aligned}
w_L &= \frac{Q_T^2}{Q^2 + Q_T^2} \, \mathcal{F}_0 \\
w_T &= \left(1 - \frac{Q_T^2}{2Q^2 + 2Q_T^2} \right) \mathcal{F}_0 \\
w_{\Delta\Delta} &= \frac{1}{2} \, \frac{Q_T^2}{\left(Q^2 + Q_T^2\right)} \, \mathcal{F}_0
\end{aligned} \tag{5.28}
$$

Now we observe here that $\mathcal{F}_0$ is the transverse structure function defined in the previous section and therefore we can apply the resummation technique used there. This allows to conclude:

$$
\boxed{
\begin{array}{lcl}
\mathcal{W}_L & = & \frac{Q_T^2}{Q^2+Q_T^2} W_T^{Total} \\
\mathcal{W}_T & = & \left[1-\frac{Q_T^2}{2Q^2+2Q_T^2}\right] W_T^{Total} \\
\mathcal{W}_{\Delta\Delta} & = & \frac{Q_T^2}{2Q^2+2Q_T^2} W_T^{Total}
\end{array}
}
$$

Table 5.1 Tensor predictions for $\mathcal{W}_L$, $\mathcal{W}_T$ and $\mathcal{W}_{\Delta\Delta}$

The new structure functions $\mathcal{W}_T$, $\mathcal{W}_L$, $\mathcal{W}_{\Delta\Delta}$ have some important properties at low Q_T. For instance, $\mathcal{W}_L$, $\mathcal{W}_{\Delta\Delta}$ α Q_T^2 which is expected from general considerations and suggested by the form of the NLO prediction for $q\bar{q}$ process, Table 3.3. $\mathcal{W}_T$ has a correction equal to $\left(1-\frac{Q_T^2}{2Q^2+2Q_T^2}\right)$ which was anticipated since the transverse structure function should decrease with an increase in Q_T, the correction should be proportional to Q_T^2 because it has to finite at $Q_T = 0$ limit.

The attentive reader should be asking now why $\mathcal{W}_\Delta$ is missing. As can be seen from Eq.(5.27) this function is proportional to $\mathcal{F}_2 - \mathcal{F}_1$. This difference is not renormalization group invariant which makes impossible the use of collinear resummation.

Despite the fact that we are working only in the CS frame the above method also is applicable to any frame. The only change necessary is to use the corresponding $\hat{z}_\mu, \hat{x}_\nu$ in the contraction with the tensor $k^{\mu\nu}$. This is an advantage if we compare with the method explained in Sec. 5.1

One observation more is important. We are neglecting here two types of terms. The first set of terms come from the finite part of the tensor separation. These terms are finite at low

Q_T limit. The second set of terms, which are proportional to a finite power of $\ln \frac{Q^2}{Q_T^2}$, come from taking only the divergent part of $\mathcal{F}_1$ and $\mathcal{F}_2$. Since both sets of terms can be neglected compared with the terms that we have retained and resummed, we expect that our results capture the relevant physics.

We present in Figures 5.5-5.7 the predicted behavior for $\mathcal{W}_L$, $\mathcal{W}_T$ and $\mathcal{W}_{\Delta\Delta}$ for $p\bar{p}$ in the Collins-Soper frame with $Q = 10\ GeV/c$, $y = 0$ and $\sqrt{S} = 800\ GeV/c$, using the same nonperturbative tensor of the last section, equations (5.11) and (5.12).

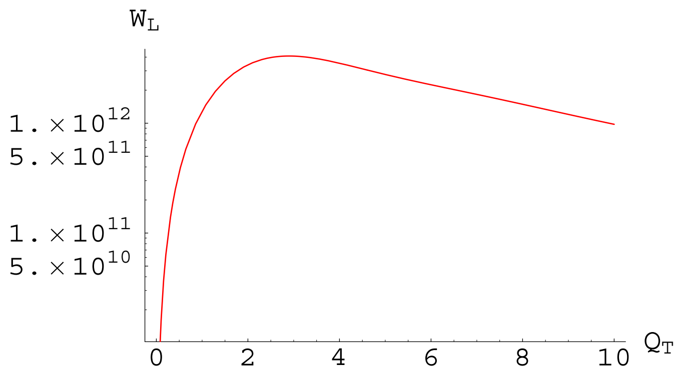


Figure 5.5 $\mathcal{W}_L$ vs Q_T extension prediction

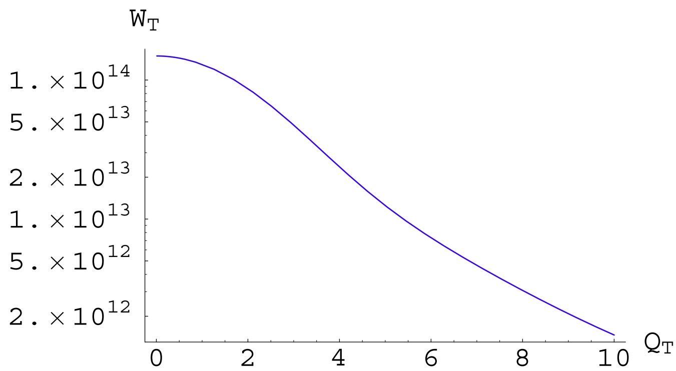


Figure 5.6 $\mathcal{W}_T$ vs Q_T extension prediction

The following step is to check for the predicted values for λ and ν which can be seen in

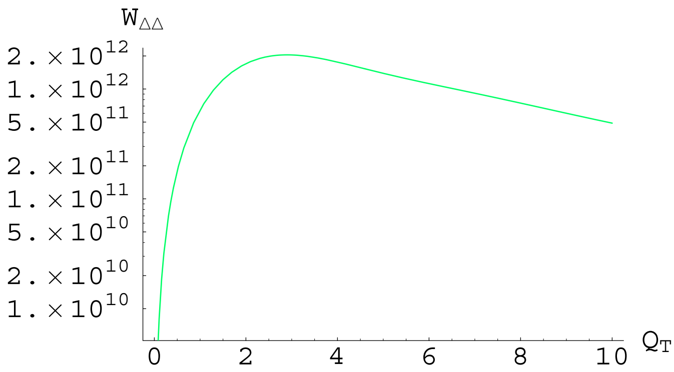


Figure 5.7 $\mathcal{W}_{\Delta\Delta}$ vs Q_T extension prediction

Table 5.2.

λ	$=$	$\frac{Q^2-\frac{1}{2}Q_T^2}{Q^2+\frac{3}{2}Q_T^2}$
ν	$=$	$\frac{Q_T^2}{Q^2+\frac{3}{2}Q_T^2}$

Table 5.2 Tensor predictions for λ and ν

Notice that we have an important relation between $\mathcal{W}_L$ and $\mathcal{W}_{\Delta\Delta}$:

$$\frac{\mathcal{W}_{\Delta\Delta}}{\mathcal{W}_L} = \frac{1}{2} \tag{5.29}$$

which represents no departure from our previous results, since Eq. (5.29) is equivalent to the Lam-Tung relation:

$$1 - \lambda - 2\nu = 0$$

This fact may explain why the violation of the LT relation at NNLO is so small. Now, compare Table 3.3 with Table 5.2. We have recovered the results of NLO with finite structure functions, thus the graphics for NLO become also the graphics for the extension of resummation.

We can observe again the behavior for λ, ν in figures 5.8 - 5.15.

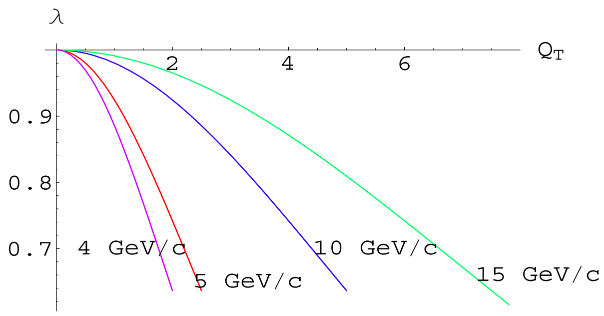


Figure 5.8 λ vs Q_T extension prediction

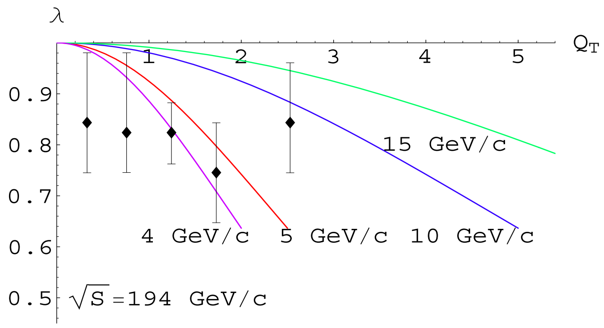


Figure 5.9 λ vs Q_T for extension and NA10

5.3 Conclusions

This study had the following objectives:

1. Extend resummation to the fully differential cross section.

2. Use the new extension to explain the azimuthal asymmetry and,

3. Comprehend the magnitude of the violation of the Lam-Tung relation and the difference in sign between the known predictions and of the measured values.

We will finish by summarizing what is has been found in each one of these goals.

- In Sec. 5.2 we developed a new method that extends resummation in the Drell-Yan process to the longitudinal and double delta structure functions. This new method also yielded a modification of the transverse structure function. This modification includes

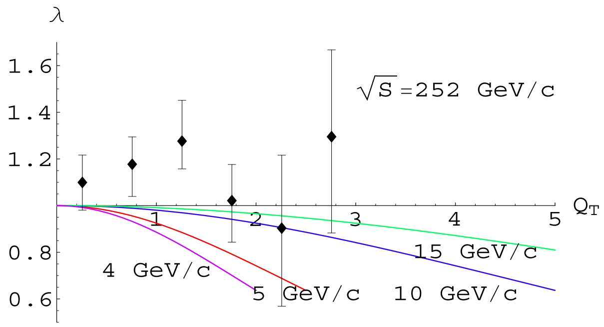


Figure 5.10 λ vs Q_T for extension and E615

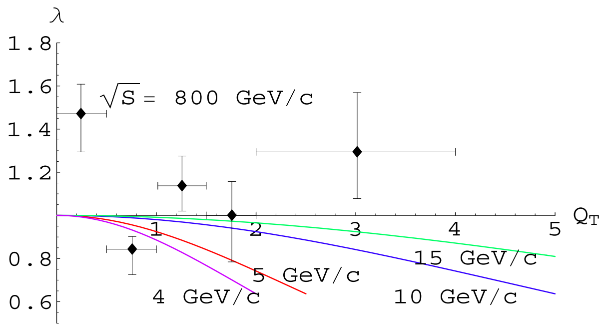


Figure 5.11 λ vs Q_T for extension and E886

now a quadratic dependence in the transverse momentum and in the invariant mass of the dilepton. This new technique also includes the effects of the nonperturbative part of the Sudakov factor.

- We have studied the structure functions W_T, W_L, $W_{\Delta\Delta}$ and W_Δ, exploring the contributions of the annihilation process $q+\bar{q} \to \gamma^* + g$ and Compton subprocess $q+g \to \gamma^* + q$. This exploration was done following the parton model, perturbative QCD, colllinear factorization and resummation and "extension" of resummation. The new functions $\mathcal{W}_T$, $\mathcal{W}_L$ and $\mathcal{W}_{\Delta\Delta}$ are finite in the low Q_T limit, which is an improvement over the NLO calculations, which diverge as a power of $1/Q_T$, see Tables 3.3, 3.5 and 3.7.

- We have also studied the angular coefficients λ, μ and ν defined on terms of the structure functions. As it can be seen in Table 5.2, the predictions obtained in the extended resummation for λ and ν are independent of the parton distribution functions and they

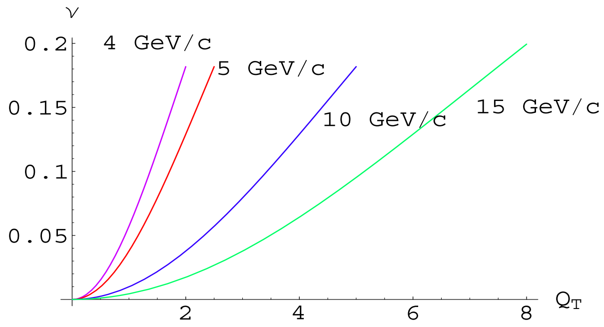


Figure 5.12 ν vs Q_T extension prediction

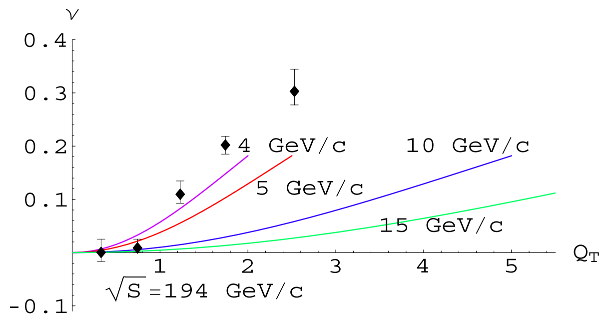


Figure 5.13 ν vs Q_T for extension and NA10

are well defined since they are ratios of finite structure functions at low Q_T. These predictions contrast with the NLO results presented in Tables 3.4, 3.6 and 3.8, where the functions obtained are ratios between divergent structure functions. This fact puts into doubt the usefulness of the NLO predictions at low Q_T. We have reproduced these results in the frame of the extended resummation.

- Experiments NA10, E615 and E866 are fixed target experiments. NA10 performed DY in $\pi^- + W$ at 194 GeV/c with kinematic variables[8]: (x_F, Q, Q_T) with ranges: $0 \leq x_F \leq 0.6$, $4.7 \leq Q \leq 8.5$ GeV/c and $Q > 11$ GeV/c [67] and [77]. E615 used also $\pi^- + W$ but at 252 GeV/c with ranges $0.2 < x_F < 1$ and $4.05 \leq Q \leq 8.55$ GeV/c [53]. E866 used $p + d$ at 800 GeV/c with ranges $0 \leq x_F \leq 0.8$, $4.5 \leq Q \leq 9$ GeV/c and $Q > 10.7$ GeV/c [128].

[8] x_F is known as the *Feynman-x variable* and it is defined as $x = \frac{P_L}{P_{L\ max}} \approx \frac{2 \sinh y \sqrt{Q^2+Q_T^2}}{\sqrt{S}}$ where P_L is the longitudinal momentum of the particle and $P_{L\ max} = \frac{\sqrt{S}}{2}$ is the maximum longitudinal momentum allowed.

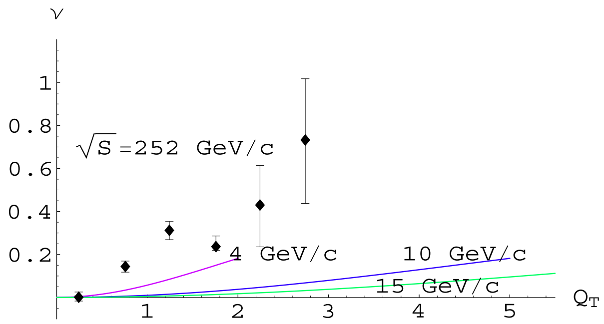


Figure 5.14 ν vs Q_T for extension and E615

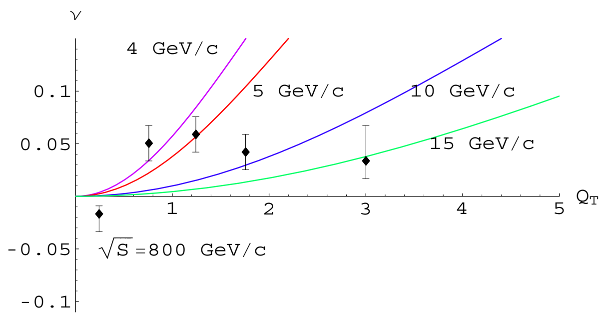


Figure 5.15 ν vs Q_T for extension and E886

These experiments divided each Q_T bin in different number of bins in $(\cos\theta, \phi)$. Then, they find the angular parameters using a standard least-squares fit to the distribution:

$$\frac{dN}{d\Omega} = \frac{3}{4\pi}\frac{1}{\lambda+3}\left(1 + \lambda\cos^2\theta + \mu\sin 2\theta\cos\phi + \frac{\nu}{2}\cos 2\phi\sin^2\theta\right)$$

This procedure presents a conceptual problem. As it can be observed in the matrix presented in Eq. (2.1), the transformation to the CS reference frame is a continuous function of the measured quantities. This in practice means that the measured angles only make sense if we keep fixed the kinematical variables because a change in them implies a change in the reference frame. Since the result presented in Figures 1.1 and 1.2 have *integrated* Q and x_F dependence there is the possibility that the observed results have integrated part of their physical meaning. An ideal experimental analysis will use bins in (x_F, Q, Q_T) as small as possible[9].

[9] Of course statistics and the particular experimental set-up will play a very important role

- The λ behavior is not clear from the experimental point of view. As it was remarked before, when we compared the NLO predictions with the data, most of the central points of the E615 and E866 sets are above 1, see Figures 5.10 and 5.11. Since $-1 \leq \lambda \leq 1$ is the allowed range, we will ignore those points. The NA10 points show a qualitative match with the theoretical curves, which is also the case of the NLO predictions, Fig. 3.6.

- A question that remains unanswered is the magnitude of ν, which gave origin to the so-called azimuthal asymmetry. The predictions found here for ν cannot explain the results reported by the NA10 and E615 collaborations because they are consistently below the experimental values, see Figures 5.13 and 5.14. This is also the case for NLO and NNLO predictions in pQCD [96]. For the E866 results Fig. 5.15 we have a qualitative agreement. We remark that the theoretical curves show a strong Q-dependence that should be included in the experimental analysis. The Q^2-dependence is supported by the data [55].

- We did not obtain, with the extended resummation, a violation of the Lam-Tung relation. The NNLO corrections only predict a minute violation [96]. At first look this will render the prediction done here useless. As it was point out previously, the central values obtained by E615 and E866 collaborations for λ are above 1 which is not physical (see Eq.(2.26)). To see see how this affects the experimental values obtained by the experiments, observe that since the Lam-Tung relation is equal to $2\nu - (1 - \lambda)$, as quoted by the experiments, we have a positive value coming from the $(1 - \lambda)$, which combined with 2ν gives a positive result. Since λ should decrease with an increase in transverse momentum, we expect $(1 - \lambda)$ to be positive and therefore to decrease the value of 2ν. From the theoretical point of view, a violation of the Lam-Tung relation reflects how fast ν increases and how fast λ decreases and the sign will depend precisely on that. A different way to see this is using the structure functions: $2\nu - (1 - \lambda) \geq 0$ is equivalent to $2W_{\Delta\Delta} \geq W_L$. The sign of the violation tells us about the relative size between these two functions. Notice here that the NA10 set is consistent with the physical expectations.

Since the experimental results have integrated the Q and x_F dependence, is difficult to assert how big or the sign of the violation. We believe that the Lam-Tung relation requires further theoretical and experimental study.

- The techniques developed here can be applied to $W^{\pm}$, Z^0 or Higgs production. These cases will require a generalization of the tensor structure used, since we will have to account for a parity violation term. This study is also relevant for processes like semi-inclusive deep inelastic scattering and back-to-back hadron production, in two-jet events in electron-positron annihilation, where no angular distributions exist. There are also plenty of possibilities for further theoretical application.

APPENDIX A. ADDITIONAL REFERENCE FRAMES

This appendix contains descriptions of some extra reference frames used in the literature. They are all expressed in terms of variables measured in the center of frame and their transformation matrices from this frame are also included.

Some dilepton center-of-mass frames

Since all the dilepton frames are related by a rotation around the common y-axis we will define the rest of the frames mentioned in Sec. 2.1 in terms of matrix of a rotation from the CS frame. The general form of this matrix is $\left(\begin{smallmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{smallmatrix}\right)$ where γ is the angle between a specified vector and the z-axis in the CS frame. With this definition the matrices of transformation from the hadron c.m.s to any particular dilepton c.m.s are easily found.

Gottfried-Jackson frame

For example, in the GJ frame the z-axis is parallel to the three-vector $\vec{P}'_A$ See Fig. A.1. The components of this vector in the CS frame are $\vec{P}'_A = \frac{\sqrt{s}}{2}\sqrt{\frac{Q_0-Q_z}{Q_0+Q_z}}\left(\frac{-Q_T}{Q},0,1\right)$, so we need to rotate an angle $\gamma_{GJ} = \arctan\left(\frac{Q_T}{Q}\right)$. The matrix of rotation is defined by the values $\cos\gamma_{GJ} = \frac{Q}{\sqrt{Q^2+Q_T^2}}$ and $\sin\gamma_{GJ} = \frac{Q_T}{\sqrt{Q^2+Q_T^2}}$. And the matrix of transformation of coordinates is

$$\Lambda_{CM\to GJ} = \begin{pmatrix} \frac{Q_0}{Q} & -\frac{Q_T}{Q} & 0 & -\frac{Q_z}{Q} \\ -\frac{Q_T}{Q_0-Q_z} & 1 & 0 & \frac{Q_T}{Q_0-Q_z} \\ 0 & 0 & 1 & 0 \\ \frac{Q_T^2 Q_0 - Q^2 Q_z}{Q\left(Q_0^2-Q_z^2\right)} & -\frac{Q_T}{Q} & 0 & \frac{Q^2 Q_0 - Q_T^2 Q_z}{Q\left(Q_0^2-Q_z^2\right)} \end{pmatrix} \tag{A.1}$$

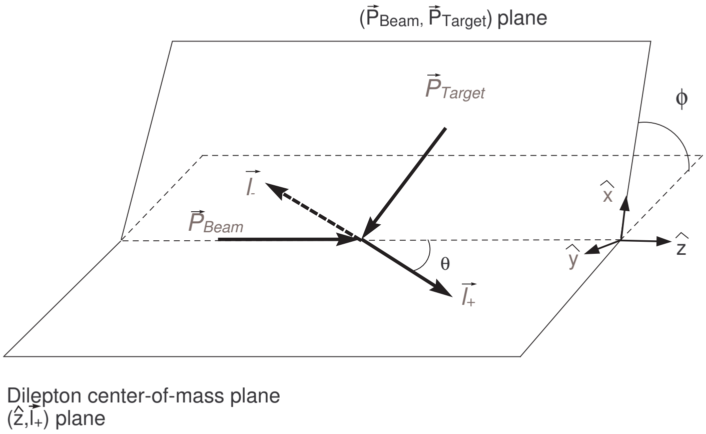


Figure A.1 The Gottfried-Jackson frame

U-channel frame

For this frame the z-axis is antiparallel to the direction of the target momentum $\vec{P}_B$, see Fig.A.2. The components of this vector are $\vec{P}_B' = \frac{\sqrt{s}}{2}\sqrt{\frac{Q_0-Q_z}{Q_0+Q_z}}\left(\frac{-Q_T}{\sqrt{Q^2+Q_T^2}}, 0, \frac{-Q}{\sqrt{Q^2+Q_T^2}}\right)$ and the angle of rotation is the same as in the GJ frame but the rotation is performed in the opposite direction. Therefore, the matrix of transformation of frames is:

$$\Lambda_{CM\to UC} = \begin{pmatrix} \frac{Q_0}{Q} & -\frac{Q_T}{Q} & 0 & -\frac{Q_z}{Q} \\ -\frac{Q_T}{Q_0+Q_z} & 1 & 0 & -\frac{Q_T}{Q_0+Q_z} \\ 0 & 0 & 1 & 0 \\ -\frac{Q_T^2 Q_0 + Q^2 Q_z}{Q\left(Q_0^2-Q_z^2\right)} & \frac{Q_T}{Q} & 0 & \frac{Q^2 Q_0 + Q_T^2 Q_z}{Q\left(Q_0^2-Q_z^2\right)} \end{pmatrix} \tag{A.2}$$

Then we can describe the z-axis in the CS frame as the bisector of the angle between the t-channel and u-channel frames [53].

S-helicity frame

The s-helicity frame is the last one to be consider here. The z-axis is antiparallel to the direction of $\vec{P}_A + \vec{P}_B$, see Fig.A.3.
This vector has components $\vec{P}_A' + \vec{P}_B' = \frac{\sqrt{s}}{Q}\sqrt{\frac{Q_T^2 Q_0^2 - Q_z^2 Q^2}{Q_0^2-Q_z^2}}\left(\frac{-Q_T Q_0}{\sqrt{Q_T^2 Q_0^2 - Q_z^2 Q^2}}, 0, \frac{-Q_z Q}{\sqrt{Q_T^2 Q_0^2 - Q_z^2 Q^2}}\right)$

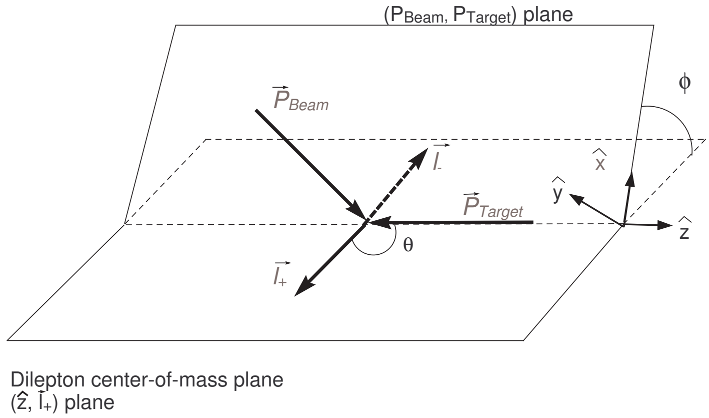


Figure A.2 The U-channel frame

Therefore, the angle of rotation is $\gamma_{SH} = \arctan\left(\frac{Q_T Q_0}{Q Q_z}\right)$, and the transformation matrix,

$$\Lambda_{CM\to SH} = \begin{pmatrix} \frac{Q_0}{Q} & -\frac{Q_T}{Q} & 0 & -\frac{Q_z}{Q} \\ 0 & \frac{Q_z}{|\vec{Q}|} & 0 & -\frac{Q_T}{|\vec{Q}|} \\ 0 & 0 & 1 & 0 \\ -\frac{|\vec{Q}|}{Q} & \frac{Q_T Q_0}{Q|\vec{Q}|} & 0 & \frac{Q_z Q_0}{Q|\vec{Q}|} \end{pmatrix} \tag{A.3}$$

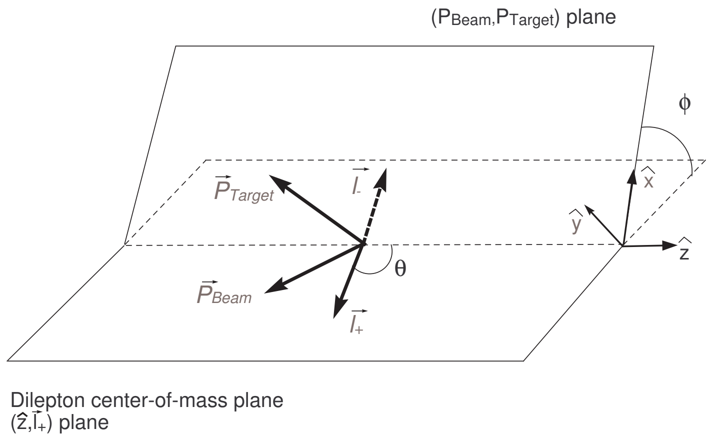


Figure A.3 The S-helicity frame

where $\left|\vec{Q}\right| = \sqrt{Q_T^2 + Q_z^2}$ is the magnitude of the vector part of Q as seen in the hadron c.m.s. The matrix A.3 can also be described as the combination of a rotation in the hadron c.m.s. and a boost. The matrix of rotation is defined by

$$\begin{pmatrix} \frac{Q_z}{\vec{Q}} & -\frac{Q_T}{|\vec{Q}|} \\ \frac{Q_T}{|\vec{Q}|} & \frac{Q_z}{|\vec{Q}|} \end{pmatrix} \tag{A.4}$$

This rotation makes the new z-axis parallel to $\vec{Q}$. In this intermediate frame Q has as components $\left(Q_0, 0, 0, \left|\vec{Q}\right|\right)$ and the boost takes this four-vector to the dilepton c.m.s. The boost matrix is equal to

$$\begin{pmatrix} \frac{Q_0}{Q} & 0 & 0 & -\frac{|\vec{Q}|}{Q} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{|\vec{Q}|}{Q} & 0 & 0 & \frac{Q_0}{Q} \end{pmatrix} \tag{A.5}$$

APPENDIX B. NLO CORRECTIONS TO THE DRELL-YAN PROCESS

The corrections of $\mathcal{O}(\alpha_S)$ for DY are presented in dimensional regularization, with $n = 4 - 2\epsilon$.

From factorization theorem to measured cross section

The factorization theorem for the DY with measured transverse momentum Eq.(3.68) can be written as:

$$\frac{d\sigma_{h_A+h_B\to l^+l^-}}{dQ^2dyd^2\vec{Q}_Td\Omega} = \sigma_0\sum_{a,b}\int_{x_A}^{1}\frac{d\xi_A}{\xi_A}\int_{x_B}^{1}\frac{d\xi_B}{\xi_B}f_{a/A}\left(\xi_A,\mu\right)f_{b/B}\left(\xi_B,\mu\right) \times T_{ab}\left(Q_T,Q,\theta,\phi,\frac{x_A}{\xi_A},\frac{x_B}{\xi_B};\mu,\alpha_S(\mu)\right) \tag{B.1}$$

where we have already used $\mu = \mu_F = Q$ and defined

$$\sigma_0 \equiv \frac{\alpha^2}{12SQ^2}$$

Our objective now is to find the NLO corrections to the naive DY. Since the hard scattering function T, but not f, has a perturbative expansion we can write:

$$T_{ab}\left(Q_T,Q,\theta,\phi,\frac{x_A}{\xi_A},\frac{x_B}{\xi_B};\mu,\alpha_S(\mu)\right) = \sum_{n=0}^{\infty}\left[\frac{\alpha_S\left(\mu\right)}{\pi}\right]^n \times T_{ab}^n\left(Q_T,Q,\theta,\phi,\frac{x_A}{\xi_A},\frac{x_B}{\xi_B};\mu\right) \tag{B.2}$$

In the lowest order of perturbation theory only $q+\bar{q}\to\gamma^*$ can contribute, so T^0 is defined by the parton cross section Eq.(3.15) and is equal to:

$$T_{ab}^0 = e_a^2\delta_{a\bar{b}}\delta\left(\frac{x_A}{\xi_A}-1\right)\delta\left(\frac{x_B}{\xi_B}-1\right)\delta^2\left(\vec{Q}_T\right)\left(1+\cos^2\theta\right) \tag{B.3}$$

For any higher order we can proceed as follows. Since the hard-scattering function is independent of the external hadrons, we can compute it using Eq.(B.1) assuming the particular case of parton-parton reaction. Then the distribution functions f now represent the parton content of the external partons and $\frac{d\sigma_{j+k\to l^+l^-}}{dQ^2dyd^2\vec{Q}_Td\Omega}$ is the $n-$dimensional scattering cross section which contains poles as $\epsilon \to 0$. Since the external particles are partons, perturbative expansions now exist for $f_{a/j}$ making possible to find T [30]. The pertubative expansion of the functions $f_{a/j}(\xi;\mu,\epsilon)$ can be obtained from their definitions (3.59) and (3.63), using the adequate parton states. In the MS scheme we can find [30], [45], [46]:

$$f_{a/j}(\xi;\mu,\epsilon) = \delta_{aj}\delta(1-\xi) - \frac{1}{2\epsilon}\frac{\alpha_S}{\pi}P^{(1)}_{a/j}(\xi) + \mathcal{O}\left(\alpha_S^2\right) \tag{B.4}$$

with $P^{(1)}_{a/j}(\xi)$ the lowest order Altarelli-Parisi kernel that provides the evolution with μ of the parton distribution functions Eq.(3.61). In Table B.1, we list the one loop kernels of QCD [102]. Once we use Eq.(B.4) into the factorization theorem we get:

$$\begin{aligned}
&\frac{d\sigma^{(0)}_{jk}}{dQ^2dyd^2\vec{Q}_Td\Omega} + \frac{\alpha_S}{\pi}\frac{d\sigma^{(1)}_{jk}}{dQ^2dyd^2\vec{Q}_Td\Omega} \\
&\qquad = \sigma_0\left\{T^{(0)}_{jk} + \frac{\alpha_S}{\pi}T^{(1)}_{jk}\right. \\
&\qquad\quad - \frac{1}{2\epsilon}\frac{\alpha_S}{\pi}\sum_a\int_{x_A}^1 d\xi_A\, P^{(1)}_{a/j}(\xi_A)\; T^{(0)}_{ak}\left(\frac{x_A}{\xi_A}, x_B, Q, \theta; \mu; \epsilon\right) \\
&\qquad\quad \left. - \frac{1}{2\epsilon}\frac{\alpha_S}{\pi}\sum_b\int_{x_B}^1 d\xi_B\, P^{(1)}_{b/k}(\xi_B)\; T^{(0)}_{jb}\left(x_A, \frac{x_B}{\xi_B}, Q, \theta; \mu; \epsilon\right)\right\} \\
&\qquad\quad + \mathcal{O}\left(\alpha_S^2\right)
\end{aligned} \tag{B.5}$$

which allows us to obtain at one loop:

$$\begin{aligned}
T^{(1)}_{jk} = \quad & \frac{1}{\sigma_0}\frac{d\sigma^{(1)}_{jk}}{dQ^2dyd^2\vec{Q}_Td\Omega} \\
+ \quad & \frac{1}{2\epsilon}\sum_a\int_{x_A}^1 d\xi_A\; P^{(1)}_{a/j}(\xi_A)\; T^{(0)}_{ak}\left(\frac{x_A}{\xi_A}, x_B, Q, \theta; \mu; \epsilon\right) \\
+ \quad & \frac{1}{2\epsilon}\sum_b\int_{x_B}^1 d\xi_B\, P^{(1)}_{b/k}(\xi_B)\; T^{(0)}_{jb}\left(x_A, \frac{x_B}{\xi_B}, Q, \theta; \mu; \epsilon\right)
\end{aligned} \tag{B.6}$$

Hence, we can obtain $T^{(1)}_{jk}$ subtracting from the parton cross section certain factors containing $1/\epsilon$, the Altarelli-Parisi kernel and the parton level result. Notice that when $\Lambda_{QCD} \ll Q \approx Q_T$ we have $T^{(1)}_{jk} = \frac{1}{\sigma_0}\frac{d\sigma^{(1)}_{jk}}{dQ^2dyd^2\vec{Q}_Td\Omega}$,which is given only by the sum of the real emission and the Compton subprocesses.

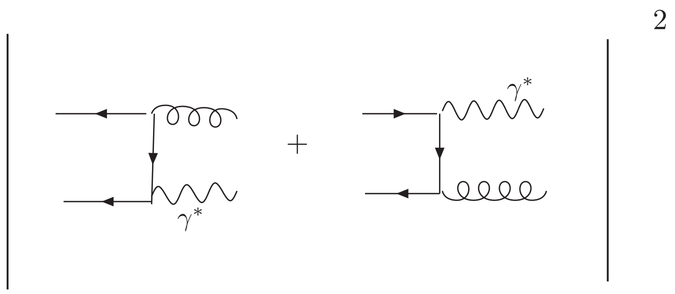
$$\begin{array}{lclcl}
P^{(1)}_{q/q}(z) & = & P^{(1)}_{\bar{q}/\bar{q}}(z) & = & \frac{4}{3}\left[(1+z^2)\frac{1}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right] \\
P^{(1)}_{\bar{q}/q}(z) & = & P^{(z)}_{q/\bar{q}}(z) & = & 0 \\
P^{(1)}_{q/g}(z) & = & P^{(1)}_{\bar{q}/g}(z) & = & \frac{1}{2}\left[z^2+(1-z)^2\right] \\
P^{(1)}_{g/q}(z) & = & P^{(1)}_{g/\bar{q}}(z) & = & \frac{4}{3}\left[\frac{1+(1+z)^2}{z}\right] \\
P^{(1)}_{g/g}(z) & = & 6\left[\frac{(1-z)}{z}\right. & + & \frac{z}{(1-z)_+}+z(1-z)+\left(\frac{11}{12}-\frac{n_f}{18}\right)\delta(1-z)\;]
\end{array}$$

Table B.1 One-loop evolution kernels in QCD

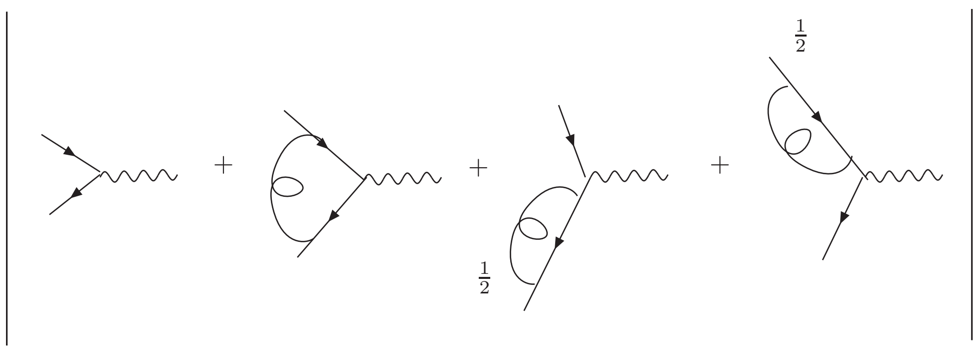
n-dimensional parton cross section at one-loop

In order to calculate the DY cross section to one-loop, we need to consider the sum of following contributions [115], [127] :

Real emission diagrams:

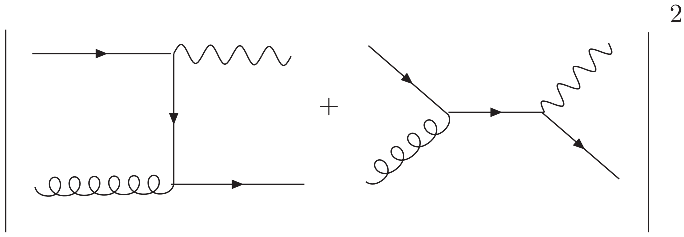


Virtual corrections diagrams:



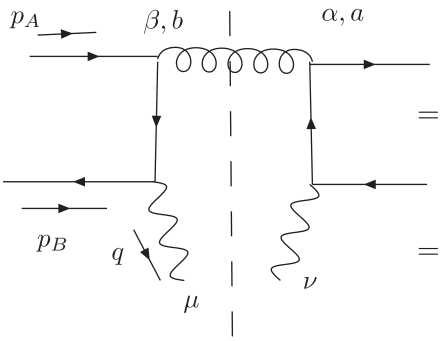
where we have included only half of the renormalization of certain fermion lines, since the other half is included in the renormalization of the parton distribution functions.

Compton subprocess:

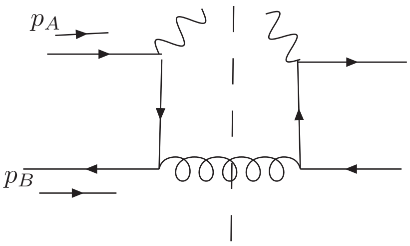


Since we are interested in the predictions of pQCD for the structure functions $W_T, W_L, W_\Delta, W_{\Delta,\Delta}$, we will keep the Lorentz indices associated with the virtual photon.

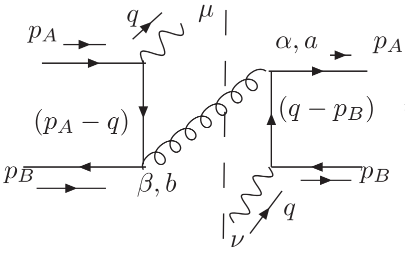
Real emission subprocess



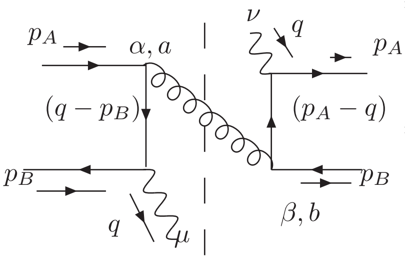
$$
\begin{aligned}
&= -\frac{\frac{1}{4}C_F\mu^{2\epsilon}}{(q-p_B)^4} Tr\left[\not{p}_A\gamma^\alpha\,(q-p_B)\cdot\gamma\;\gamma^\nu\not{p}_B\gamma^\mu\,(q-p_B)\cdot\gamma\;\gamma_\alpha\right] \\
&= (1-\epsilon)\,\frac{C_F\mu^{2\epsilon}}{t}\,[\quad -\quad ug^{\mu\nu} + 2\left(p_B^\mu q^\nu + p_B^\nu q^\mu\right) \\
&\qquad\qquad\qquad\quad -\quad 4p_B^\mu p_B^\nu - 2\left(p_A^\mu p_B^\nu + p_A^\nu p_B^\mu\right)\;]
\end{aligned}
$$



$$
\begin{aligned}
&= (1-\epsilon)\,\frac{C_F\mu^{2\epsilon}}{u}\,[\quad -\quad tg^{\mu\nu} + 2\left(p_A^\mu q^\nu + p_A^\nu q^\mu\right) \\
&\qquad\qquad\qquad\quad -\quad 4p_A^\mu p_A^\nu - 2\left(p_B^\mu p_A^\nu + p_B^\nu p_A^\mu\right)\;]
\end{aligned}
$$



$$
\begin{aligned}
&= -\frac{\frac{1}{4}C_F\mu^{2\epsilon}}{tu} Tr\left[\not p_A \gamma^\alpha \left(q - p_B\right)\cdot\gamma\, \gamma^\nu \not p_B \gamma_\alpha \left(p_A - q\right)\cdot\gamma\, \gamma^\mu\right] \\
&= \frac{C_F\mu^{2\epsilon}}{ut}\{ \quad - \quad 2sp_A^\mu p_B^\nu + 2sq^\mu p_B^\nu - 2\left(Q^2 - u\right) p_B^\mu p_B^\nu + 2sp_A^\mu q^\nu \\
&\qquad\qquad - \quad 2\left(Q^2 - t\right) p_A^\mu p_A^\nu + 2Q^2 p_B^\mu p_A^\nu - sQ^2 g^{\mu\nu} \\
&\qquad\qquad - \quad \epsilon\left[2sp_A^\mu q^\nu - 2sp_A^\mu p_B^\nu - 2\left(Q^2 - t\right) p_A^\mu p_A^\nu\right. \\
&\qquad\qquad + \quad 2\left(Q^2 - t\right) q^\mu p_A^\nu - 2sq^\mu q^\nu + 2sq^\mu p_B^\nu - 2Q^2 p_B^\mu p_A^\nu \\
&\qquad\qquad + \quad \left. 2\left(Q^2 - u\right) p_B^\mu q^\nu - 2\left(Q^2 - u\right) p_B^\mu p_B^\nu - utg^{\mu\nu} \;\right]\}
\end{aligned}
$$



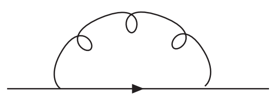
$$
\begin{aligned}
&= -\frac{\frac{1}{4}C_F\mu^{2\epsilon}}{tu} Tr\left[\not p_A \gamma^\nu \left(p_A - q\right)\cdot\gamma\, \gamma^\alpha \not p_B\, \gamma^\mu \left(q - p_B\right)\cdot\gamma\, \gamma_\alpha\right] \\
&= \frac{C_F\mu^{2\epsilon}}{ut}\{ \quad - \quad sQ^2 g^{\mu\nu} - 2\left(Q^2 - t\right) p_A^\mu p_A^\nu + 2Q^2 p_A^\mu p_B^\nu - 2sp_B^\mu p_A^\nu \\
&\qquad\qquad + \quad 2sq^\mu p_A^\nu - 2\left(Q^2 - u\right) p_B^\mu p_B^\nu + 2sp_B^\mu q^\nu \\
&\qquad\qquad - \quad \epsilon\left[-2sp_B^\mu p_A^\nu + 2sp_B^\mu q^\nu - 2Q^2 p_A^\mu p_B^\nu\right. \\
&\qquad\qquad + \quad 2\left(Q^2 - u\right) q^\mu p_B^\nu - utg^{\mu\nu} + 2sq^\mu p_A^\nu \\
&\qquad\qquad + \quad 2\left(Q^2 - t\right) p_A^\mu q^\nu - 2\left(Q^2 - t\right) p_A^\mu p_A^\nu \\
&\qquad\qquad - \quad \left. 2sq^\mu q^\nu - 2\left(Q^2 - u\right) p_B^\mu p_B^\nu \;\right]\}
\end{aligned}
$$

The color factor C_F is equal to $\frac{1}{9}\sum_{a,b=1}^{8} Tr\left[\lambda^a\lambda^b\right] = \frac{4}{9}$, where we have used the standard normalization for the color matrices $Tr\left[\lambda^a\lambda^b\right] = \frac{1}{2}\delta^{ab}$. Finally, we can add our previous results to obtain for $q\bar{q}$:

$$
\begin{aligned}
h_{j\bar{j}}^{\mu\nu(R)} = \frac{4}{9}\frac{e_j^2 g^2\mu^{2\epsilon}}{ut}\{ \quad & - \quad 4Q^2\left(p_A^\mu p_A^\nu + p_B^\mu p_B^\nu\right) - \left[\left(Q^2 - t\right)^2 + \left(Q^2 - u\right)^2\right] g^{\mu\nu} \\
& + \quad 2\left(Q^2 - t\right)\left(p_B^\mu q^\nu + p_B^\nu q^\mu\right) + 2\left(Q^2 - u\right)\left(p_A^\mu q^\nu + p_A^\nu q^\mu\right) \\
& - \quad \epsilon\left[2\left(Q^2 + s\right)\left(p_B^\mu q^\nu + q^\mu p_B^\nu\right) + 2\left(Q^2 + s\right)\left(p_A^\mu q^\nu + q^\mu p_A^\nu\right)\right. \\
& - \quad \left. 4Q^2\left(p_A^\mu p_A^\nu + p_B^\mu p_B^\nu\right) - 4Q^2\left(p_A^\mu p_B^\nu + p_B^\mu p_A^\nu\right) - \left(Q^2 - s\right)^2 g^{\mu\nu} - 4sq^\mu q^\nu \;\right]\}
\end{aligned}
\tag{B.7}
$$

with $s = \left(p_B + p_A\right)^2$, $t = \left(p_B - q\right)^2$ and $u = \left(p_A - q\right)^2$.

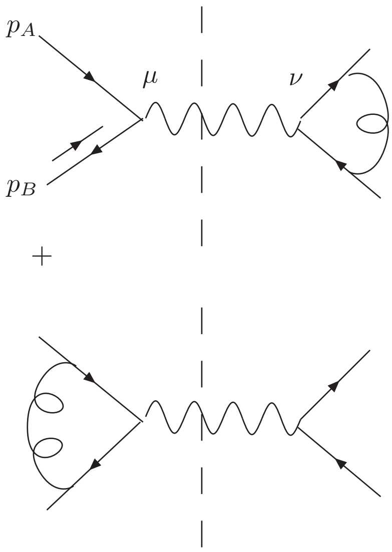
Virtual corrections



$= 0$

The virtual corrections include fermion self-energy and vertex correction diagrams . Since the basic interaction is electromagnetic these corrections have the same group structure and their corresponding counterterms cancel each other. Notice also that because the fermions are massless, the fermion self-energy without counterterm is a scaleless integral, which is equal to zero in dimensional regularization [115].

Thus, we are left only with the diagrams corresponding to vertex correction calculated as if there is not courterterm.



Since the sum of these corrections is proportional to the Born level diagram, we can write:

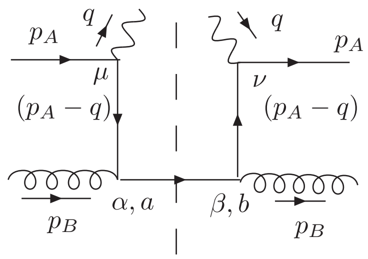
$$\begin{aligned} h^{\mu\nu(V)}_{j\bar{j}} &= \frac{1}{3}\frac{e_j^2}{4} 2Re\left(\gamma\left(Q^2\right)\right) Tr\left[\not{p}_A \gamma^\nu \not{p}_B \gamma^\mu\right] \\ &= \frac{2e_j^2}{3} Re\,\gamma\left(Q^2\right)\left(p_A^\mu p_B^\nu - \frac{s}{2} g^{\mu\nu} + p_B^\mu p_A^\nu\right) \end{aligned} \tag{B.8}$$

where [115]:

$$\begin{aligned} \gamma\left(Q^2\right) = - \frac{\alpha_S}{2\pi} \frac{4}{3}\left[\frac{4\pi\mu^2}{-Q^2}\right]^\epsilon \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \\ \times \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 4\right) \end{aligned} \tag{B.9}$$

Note that because Eq.(B.8) is not the square of an amplitude, it is not required to be positive.

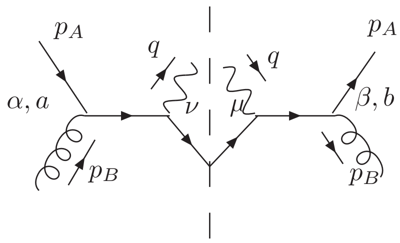
Compton subprocess



$$= -\frac{C_{F_2}\mu^{2\epsilon}}{u^2 4\,(1-\epsilon)} Tr\,[\quad \not{p}_A \quad \gamma^\mu\,(p_A - q)\cdot\gamma\;\gamma^\alpha$$

$$\times \quad (p_B + p_A - q)\cdot\gamma\;\gamma_\alpha\,(p_A - q)\cdot\gamma\;\gamma^\nu\;]$$

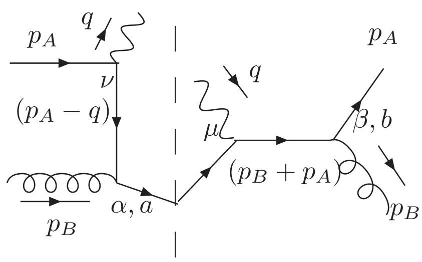
$$= \frac{C_{F_2}\mu^{2\epsilon}\,(1-\epsilon)}{u\,(1-\epsilon)}\left[sg^{\mu\nu} - 2\left(p_A^\mu p_B^\nu + p_B^\mu p_A^\nu\right)\right]$$



$$= -\frac{C_{F_2}\mu^{2\epsilon}}{s^2 4\,(1-\epsilon)} Tr\,[\quad \not{p}_A \quad \gamma^\alpha\,(p_A + p_B)\cdot\gamma\;\gamma^\mu$$

$$\times \quad (p_B + p_A - q)\cdot\gamma\;\gamma^\nu\,(p_A + p_B)\cdot\gamma\;\gamma_\alpha\;]$$

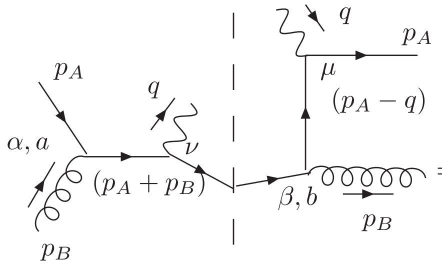
$$= \frac{(1-\epsilon)}{s\,(1-\epsilon)}\left[ug^{\mu\nu} + 2\left(p_A^\mu p_B^\nu + p_B^\mu p_A^\nu\right) + 4p_B^\mu p_B^\nu - 2\left(q^\mu p_B^\nu + p_B^\mu q^\nu\right)\right]$$



$$= -\frac{C_{F_2}\mu^{2\epsilon}}{4su\,(1-\epsilon)} Tr\,[\quad \not{p}_A \quad \gamma^\alpha\,(p_A + p_B)\cdot\gamma\;\gamma^\mu$$

$$\times \quad (p_A + p_B - q)\cdot\gamma\;\gamma_\alpha\,(p_A - q)\cdot\gamma\;\gamma^\nu\;]$$

$$\begin{aligned}
= \frac{C_{F_2}\mu^{2\epsilon}}{su\,(1-\epsilon)}\Big\{ \quad & tQ^2 g^{\mu\nu} + 4Q^2 p_A^\mu p_A^\nu + 2\left(2Q^2 - u\right) p_A^\mu p_B^\nu \\
+ \quad & 2sp_B^\mu p_A^\nu + 2\left(Q^2 - u\right) p_B^\mu p_B^\nu \\
- \quad & 2\left(Q^2 + s\right) p_A^\mu q^\nu - 2sp_B^\mu q^\nu - 2\left(Q^2 - u\right) q^\mu p_A^\nu \\
- \quad & 2\left(Q^2 - u\right) q^\mu p_B^\nu + 2sq^\mu q^\nu \\
- \quad & \epsilon\left[sug^{\mu\nu} + 2sp_B^\mu p_A^\nu - 2up_A^\mu p_B^\nu\right. \\
+ \quad & \left. 2\left(Q^2 - u\right) p_B^\mu p_B^\nu - 2sp_B^\mu q^\nu\;\right]\Big\}
\end{aligned}$$



$$
\begin{aligned}
&= -\frac{C_{F_2}\mu^{2\epsilon}}{4su\,(1-\epsilon)} Tr\,[\quad \not{p}_A \quad \gamma^\mu \,(p_A - q)\cdot\gamma\,\gamma^\alpha \\
&\qquad \times \quad (p_A + p_B - q)\cdot\gamma\,\gamma^\nu\,(p_A + p_B)\cdot\gamma\,\gamma_\alpha\,]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{C_{F_2}\mu^{2\epsilon}}{su\,(1-\epsilon)}\{ \quad & tQ^2 g^{\mu\nu} + 4Q^2 p_A^\mu p_A^\nu + 2\left(2Q^2 - u\right) p_B^\mu p_A^\nu \\
- \quad & 2Q^2 p_A^\mu p_B^\nu - 2\left(Q^2 + s\right) q^\mu p_A^\nu \\
+ \quad & 2\left(Q^2 + s\right) p_A^\mu p_B^\nu + 2\left(Q^2 - u\right) p_B^\mu p_B^\nu \\
- \quad & 2sq^\mu p_B^\nu - 2\left(Q^2 - u\right) p_A^\mu q^\nu \\
- \quad & 2\left(Q^2 - u\right) p_B^\mu q^\nu + 2sq^\mu q^\nu \\
- \quad & \epsilon\left[sug^{\mu\nu} - 2up_B^\mu p_A^\nu + 2sp_A^\mu p_B^\nu\right. \\
+ \quad & \left. 2\left(Q^2 - u\right) p_B^\mu p_B^\nu - 2sq^\mu p_B^\nu \;\right]\}
\end{aligned}
$$

Remember that in $n = 4 - 2\epsilon$ dimensions, the polarization degree of freedom is 2 for a quark and $n - 2 = 2(1 - \epsilon)$ for a gluon. This is why we have the extra factor $(1 - \epsilon)$ dividing the expressions of the previous diagrams. We have also $C_{F_2} = \frac{1}{3}\frac{1}{8}\sum_{a,b} Tr\left[\lambda^a\lambda^b\right] = \frac{1}{6}$, so the final result for the Compton subprocess qg is:

$$
\begin{aligned}
h_{jg}^{\mu\nu} = \frac{1}{6}\frac{e_j^2 g^2 \mu^{2\epsilon}}{us\,(1-\epsilon)}\{ \quad & 8Q^2 p_A^\mu p_A^\nu + 4Q^2 p_B^\mu p_B^\nu + 4Q^2\left(p_A^\mu p_B^\nu + p_B^\mu p_A^\nu\right) \\
+ \quad & \left(2tQ^2 + u^2 + s^2\right) g^{\mu\nu} - 2\left(Q^2 + t + 2s\right)\left(p_A^\mu q^\nu + q^\mu p_A^\nu\right) \\
- \quad & 2\left(Q^2 + s\right)\left(p_B^\mu q^\nu + q^\mu p_B^\nu\right) + 4sq^\mu q^\nu - \epsilon\,[\, - 2\,(u + s)\left(p_B^\mu q^\nu + q^\mu p_B^\nu\right) \\
+ \quad & \left(Q^2 - t\right)^2 g^{\mu\nu} + 4Q^2 p_B^\mu p_B^\nu \;]\}
\end{aligned}
\tag{B.10}
$$

In order to consider the diagrams with p_A and p_B exchanged, we only need to transform the above expression using $p_A \leftrightarrow p_B$ and $u \leftrightarrow t$ and thus the result for the Compton subprocess gq is:

$$
\begin{aligned}
h_{gj}^{\mu\nu} = \frac{1}{6}\frac{e_j^2 g^2 \mu^{2\epsilon}}{ts\,(1-\epsilon)}\,\{ & \quad 8Q^2 p_B^\mu p_B^\nu + 4Q^2 p_A^\mu p_A^\nu + 4Q^2 \left(p_B^\mu p_A^\nu + p_A^\mu p_B^\nu\right) \\
+ & \quad \left(2uQ^2 + t^2 + s^2\right) g^{\mu\nu} - 2\left(Q^2 + u + 2s\right)\left(p_B^\mu q^\nu + q^\mu p_B^\nu\right) \\
- & \quad 2\left(Q^2 + s\right)\left(p_A^\mu q^\nu + q^\mu p_A^\nu\right) + 4sq^\mu q^\nu - \epsilon\,[\,-2\,(t+s)\left(p_A^\mu q^\nu + q^\mu p_A^\nu\right) \\
+ & \quad \left(Q^2 - u\right)^2 g^{\mu\nu} + 4Q^2 p_A^\mu p_A^\nu\ ]\}
\end{aligned} \tag{B.11}
$$

APPENDIX C. SOME USEFUL MATHEMATICAL RESULS

We include here some mathematical results used in chapter 4.

Extraction of divergent contributions

In Eq.(4.2) a modified version of the formal identity [82]:

$$\delta\left(mn-c\right)=\frac{\delta\left(m\right)}{n_{+}}-\delta\left(m\right)\delta\left(n\right)\ln\left(c\right)+\frac{\delta\left(n\right)}{m_{+}}+\mathcal{O}(c) \tag{C.1}$$

was introduced. Here, we will explain the extra term present and the unusual definition used for the "+ distributions." We need to start with an integral equal to Eq.(4.1):

$$\int_{x_A}^{1} d\xi_A f(\xi_A)\int_{x_B}^{1} d\xi_B\, g(\xi_B)\delta\left[\left(\xi_A-x_A\right)\left(\xi_B-x_B\right)-x_Ax_B\frac{Q_T^2}{Q^2}\right]$$

introducing the change of variables $m=\xi_A-x_A$ and $n=\xi_B-x_B$ we have a new integral

$$\int_0^{1-x_A} dm\, f(m+x_A)\int_0^{1-x_B} dn\, g(n+x_B)\,\delta\left(mn-c\right)$$

with $c=x_Ax_B\frac{Q_T^2}{Q^2}$. We now take the limit of low Q_T or equivalently $c\to 0$ to obtain:

$$\lim_{c\to 0}\left[\int_{\sqrt{c}}^{1-x_A}\frac{dm}{m}f(m+x_A)\int_{\sqrt{c}}^{1-x_B} dn\, g(n+x_B)\,\delta(n)\right.$$
$$\left.+\int_{\sqrt{c}}^{1-x_A} dm\, f(m+x_A)\,\delta(m)\int_{\sqrt{c}}^{1-x_B}\frac{dn}{n}g(n+x_B)\right]$$

which is equal to

$$\lim_{c\to 0}\left[g(x_B)\int_{\sqrt{c}}^{1-x_A}\frac{dm}{m}f(m+x_A)+f(x_A)\int_{\sqrt{c}}^{1-x_B}\frac{dn}{n}g(n+x_B)\right]$$

where we have used the regions of integration defined in Fig.C.1. Taking the limit we find

$$g(x_B)\int_0^{1-x_A}\frac{dm}{m_+}f(m+x_A)+g(x_B)f(x_A)\ln\left[\frac{(1-x_A)(1-x_B)}{c}\right]+f(x_A)\int_{\sqrt{c}}^{1-x_B}\frac{dn}{n_+}g(n+x_B)$$

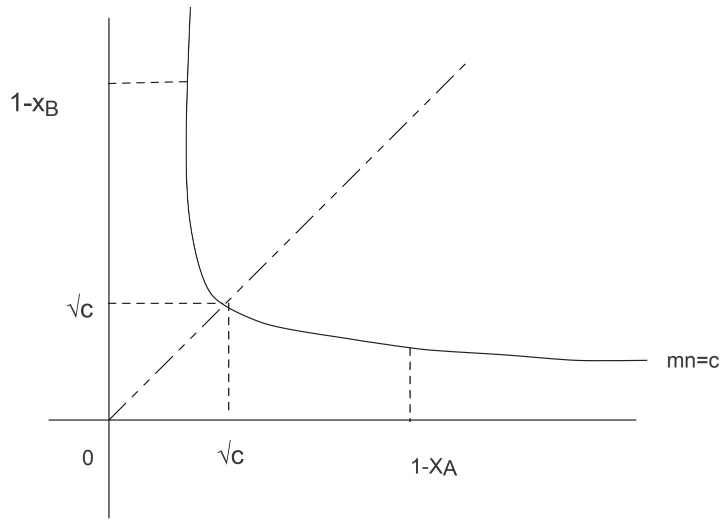


Figure C.1 Regions of integration

where the following definition

$$\int_0^b dx \frac{f(x)}{x_+} \equiv \lim_{\epsilon\to 0}\int_\epsilon^b dx \frac{f(x)-f(0)}{x} \tag{C.2}$$

$$= \lim_{\epsilon\to 0}\left[\int_\epsilon^b dx \frac{f(x)}{x} - f(0)\ln\left(\frac{b}{\epsilon}\right)\right] \tag{C.3}$$

was employed. Now, going back to the original variables we get:

$$g(x_B)\int_{x_A}^1 d\xi_A \frac{f(\xi_A)}{(\xi_A - x_A)_+} + g(x_B)f(x_A)\ln\left[\frac{(1-x_A)(1-x_B)}{c}\right] + f(x_A)\int_{x_B}^1 d\xi_B \frac{g(\xi_B)}{(\xi_B - x_B)_+} \tag{C.4}$$

The integrals containing "+ distributions" are finite and the only divergence is in the term that contains the logarithm.

Relation between plus-distributions and plus-distributions

The standard definition for the plus-distribution is [102]:

$$\int_0^1 dx\ F_+(x)G(x) \equiv \int_0^1 dx\ F(x)\left[G(x) - G(1)\right] \tag{C.5}$$

A second definition is also important [44]:

$$\int_a^1 dx\ F_+(x)G(x) \equiv \int_a^1 dx\ F(x)\left[G(x) - G(1)\right] - G(1)\int_0^a dx\ F(x) \tag{C.6}$$

Note that this definitions assume that the singularity is in $x = 1$.

Now, in Chapter 4 we have the following integral in Eq.(4.2):

$$\int_x^1 \frac{d\xi}{\xi} \frac{f(\xi)}{(\xi - x)_+} \frac{\xi^2 + x^2}{\xi} = \int_x^1 \frac{d\xi}{(\xi - x)} \left[f(\xi) \frac{\xi^2 + x^2}{\xi^2} - 2f(x) \right] \tag{C.7}$$

where the equality comes from Eq.(C.3). Observe that the singularity is in $\xi = x$, in the lower limit of the integral. In order to compare with the standard literature we need to introduce the variable $z = x/\xi$ and the definition of Eq.(C.6). So Eq.(C.7) becomes

$$\begin{aligned}
&\int_x^1 \frac{dz}{z} f(\frac{x}{z}) \frac{1+z^2}{1-z} - 2f(x) \int_x^1 \frac{dz}{z} \left(\frac{1}{1-z} \right) \\
&= \int_x^1 dz \, \frac{1+z^2}{1-z} \left[\frac{f(z)}{z} - f(x) \right] + f(x) \int_x^1 dz \, \frac{1+z^2}{1-z} - 2f(x) \int_x^1 \frac{dz}{z} \left(\frac{1}{1-z} \right) \\
&= \int_x^1 \frac{dz}{z} \left[\frac{1+z^2}{1-z} \right]_+ f(z) + f(x) \int_0^1 dz \, \frac{1+z^2}{1-z} - 2f(x) \int_x^1 \frac{dz}{z} \left(\frac{1}{1-z} \right)
\end{aligned}$$

which allows us to write

$$\int_x^1 \frac{d\xi}{\xi} \frac{f(\xi)}{(\xi - x)_+} \frac{\xi^2 + x^2}{\xi} = \int_x^1 \frac{dz}{z} \left[\frac{1+z^2}{1-z} \right]_+ f(z) - f(x) \left[\frac{3}{2} + 2 \ln \left(\frac{1-x}{x} \right) \right] \tag{C.8}$$

Using relation (C.6), we can prove the equality between the two expressions used in this thesis for the DGLAP kernel:

$$\left(\frac{1+z^2}{1-z} \right)_+ = \frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \tag{C.9}$$

The procedure used to prove the above equation is the familiar one: we introduce a test function f and we check the equality between integrals. Starting from the left hand side of Eq.(C.9):

$$\begin{aligned}
&\int_a^1 dz \, [f(z) - f(1)] \left(\frac{1+z^2}{1-z} \right) - f(1) \int_0^a dz \left(\frac{1+z^2}{1-z} \right) \\
&= \int_a^1 dz \frac{f(z)(1+z^2) - 2f(1)}{1-z} - f(1) \int_0^1 dz \left(\frac{1+z^2}{1-z} \right) + 2f(1) \int_a^1 \frac{1}{1-z} \\
&= \int_a^1 dz \frac{f(z)(1+z^2) - 2f(1)}{1-z} - 2f(1) \int_0^a \frac{1}{1-z} + \frac{3}{2} f(1) \\
&= \frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z)
\end{aligned}$$

Fourier transform in n-dimensions

For completeness reasons, we describe here the procedure outlined in [73] to perform the Fourier integral for spherical functions. A function $f(x_1, x_2, ..., x_n)$ is spherical if it depends only on the single variable $x = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}$.

Before we proceed, we need to state a few facts about ultraspherical of Gegenbauer polynomials $C_n^\alpha(x)$ [1]. The first relation that we need is their normalization conditions:

$$\int_{-1}^{1} dx\,\left(1-x^2\right)^{\alpha-1/2} C_n^\alpha(x) C_m^\alpha(x) = \delta_{mn} \frac{\pi\, 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)\,[\Gamma(\alpha)]^2} \qquad \alpha \neq 0 \tag{C.10}$$

$$= \delta_{mn}\frac{2\pi}{n^2} \qquad \alpha = 0 \tag{C.11}$$

where α is arbitrary except for the condition $\alpha > -\frac{1}{2}$. Another important property is

$$C_0^\alpha(x) = 1 \tag{C.12}$$

we will make use also of the following expansion:

$$e^{ipx\cos\theta} = \Gamma(\alpha)\left(\frac{px}{2}\right)^{-\alpha} \sum_{k=0}^{\infty} (\alpha+k) i^k J_{\alpha+k}(px) C_k^\alpha(\cos\theta) \tag{C.13}$$

where α is arbitrary and $J_\nu(x)$ is the Bessel function of first kind [1]. Now we are ready. We want to evaluate

$$\int d^n x\, f(x) e^{i\vec{p}\cdot\vec{x}} \tag{C.14}$$

with $\vec{p} = (p_1, p_2, ..., p_n)$. Using expansion Eq.(C.13) in Eq.(C.14) we find

$$\sum_{k=0}^{\infty} (\alpha+k) i^k \int dx\, x^{n-1} f(x) J_{\alpha+k}(px) \left(\frac{px}{2}\right)^{-\alpha} \int_0^\pi d\theta_{n-2} \sin^{n-2}(\theta_{n-2}) C_k^\alpha(\cos\theta_{n-2}) \int d\Omega_{n-3} \tag{C.15}$$

We can perform the integral with respect to θ_{n-2} with the help of the orthonormality relation (C.11). Comparing, we need $k = 0$ and $\alpha = \frac{n}{2} - 1$, thus we get

$$(\frac{2\pi}{p})^{\frac{n}{2}} p \int dx\, f(x) x^{\frac{n}{2}} J_{\frac{n}{2}-1}(px) \tag{C.16}$$

choosing $f(x) = \frac{1}{x^2}$, we can perform the integral after reading it in a table of integrals [72]:

$$(\frac{2\pi}{p})^{\frac{n}{2}} p \int dx\, x^{\frac{n}{2}-2} J_{\frac{n}{2}-1}(px) = \left(\frac{2}{p}\right)^{n-2} \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}-1\right) \tag{C.17}$$

likewise with $f(x) = \frac{1}{x^2}\ln(x^2)$

$$(\frac{2\pi}{p})^{\frac{n}{2}} p \int dx\, x^{\frac{n}{2}-2} J_{\frac{n}{2}-1}(px)\ln(x^2) = \left(\frac{2}{p}\right)^{n-2} \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}-1\right)\left[\psi\left(\frac{n}{2}-1\right) - \gamma_E - \ln\left(\frac{b^2}{4}\right)\right] \tag{C.18}$$

BIBLIOGRAPHY

[1] M. Abramowitz and I. Stegun, Editors "Handbook of Mathematical Functions", Ninth Printing. *New York, USA: Dover Publications, Inc., (1970) 1046p*

[2] G. Altarelli, R. K. Ellis, M. Greco and G. Martinelli, Nucl. Phys. B **246**, 12 (1984).

[3] D. Amati, R. Petronzio and G. Veneziano, Nucl. Phys. B **140**, 54 (1978).

[4] D. Amati, R. Petronzio and G. Veneziano, "Relating Hard QCD Processes Through Universality Of Mass Singularities. Nucl. Phys. B **146**, 29 (1978).

[5] Arfken, G. B. and Weber, H. J., "Mathematical Methods for Physicists, 4th ed," *London, UK: Academic Press (1995) 1029p*

[6] E. N. Argyres and C. S. Lam, Phys. Rev. D **26**, 114 (1982).

[7] P. B. Arnold and R. P. Kauffman, Nucl. Phys. B **349**, 381 (1991).

[8] J. J. Aubert *et al.* [E598 Collaboration], Phys. Rev. Lett. **33**, 1404 (1974).

[9] J. E. Augustin *et al.* [SLAC-SP-017 Collaboration], Phys. Rev. Lett. **33**, 1406 (1974).

[10] J. Badier *et al.* [Saclay-CERN-College de France-Ecole Poly-Orsay Collaboration], Phys. Lett. B **96**, 422 (1980).

[11] J. Badier *et al.* [Saclay-CERN-College de France-Ecole Poly-Orsay Collaboration], Phys. Lett. B **93**, 354 (1980).

[12] C. Balazs, J. w. Qiu and C. P. Yuan, Phys. Lett. B **355**, 548 (1995) [arXiv:hep-ph/9505203].

[13] C. Balazs and C. P. Yuan, Phys. Rev. D **56**, 5558 (1997) [arXiv:hep-ph/9704258].

[14] V. Barone, A. Drago and P. G. Ratcliffe, Phys. Rept. **359**, 1 (2002) [arXiv:hep-ph/0104283].

[15] E. L. Berger and S. J. Brodsky, Phys. Rev. Lett. **42**, 940 (1979).

[16] E. L. Berger, Phys. Lett. B **89**, 241 (1980).

[17] E. L. Berger, arXiv:hep-ph/9801381.

[18] E. D. Bloom *et al.*, Phys. Rev. Lett. **23**, 930 (1969).

[19] D. Boer and P. J. Mulders, Phys. Rev. D **57**, 5780 (1998) [arXiv:hep-ph/9711485].

[20] D. Boer, Phys. Rev. D **60**, 014012 (1999) [arXiv:hep-ph/9902255].

[21] D. Boer, Nucl. Phys. B **603**, 195 (2001) [arXiv:hep-ph/0102071].

[22] D. Boer, S. J. Brodsky and D. S. Hwang, Phys. Rev. D **67**, 054003 (2003) [arXiv:hep-ph/0211110].

[23] D. Boer, A. Brandenburg, O. Nachtmann and A. Utermann, Eur. Phys. J. C **40**, 55 (2005) [arXiv:hep-ph/0411068].

[24] D. Boer
"Sudakov suppression of asymmetries involving TMDs,"
`http://quark.phy.bnl.gov/~fyuan/workshop/Boer.pdf`
Accessed, June 08, 2006

[25] D. Boer and W. Vogelsang, arXiv:hep-ph/0604177.

[26] G. Bozzi, S. Catani, D. de Florian and M. Grazzini, Phys. Lett. B **564**, 65 (2003) [arXiv:hep-ph/0302104].

[27] A. Brandenburg, O. Nachtmann and E. Mirkes, Z. Phys. C **60**, 697 (1993).

[28] A. Brandenburg, S. J. Brodsky, V. V. Khoze and D. Mueller, Phys. Rev. Lett. **73**, 939 (1994) [arXiv:hep-ph/9403361].

[29] S. J. Brodsky, D. S. Hwang and I. Schmidt, Phys. Lett. B **530**, 99 (2002) [arXiv:hep-ph/0201296].

[30] R. Brock *et al.* [CTEQ Collaboration], "Handbook of perturbative QCD: Version 1.0," Rev. Mod. Phys. **67**, 157 (1995).

[31] C. G. . Callan and D. J. Gross, Phys. Rev. Lett. **21**, 311 (1968)

[32] C. G. . Callan and D. J. Gross, Phys. Rev. Lett. **22**, 156 (1969).

[33] P. Chiappetta and M. Le Bellac, Z. Phys. C **32**, 521 (1986).

[34] J. H. Christenson, G. S. Hicks, L. M. Lederman, P. J. Limon, B. G. Pope and E. Zavattini, Phys. Rev. Lett. **25**, 1523 (1970).

[35] J. H. Christenson, G. S. Hicks, L. M. Lederman, P. J. Limon, B. G. Pope and E. Zavattini, Phys. Rev. D **8**, 2016 (1972).

[36] Clay Mathematics Institute
`http://www.claymath.org/millennium/Yang-Mills_Theory/`
Accessed, August 11, 2006

[37] J. Cleymans and M. Kuroda, Nucl. Phys. B **155**, 480 (1979) [Erratum-ibid. B **160**, 510 (1979)].

[38] J. Cleymans and M. Kuroda, Phys. Lett. B **105**, 68 (1981).

[39] J. C. Collins and D. E. Soper, Phys. Rev. D **16**, 2219 (1977).

[40] J. C. Collins, Phys. Rev. Lett. **42**, 291 (1979).

[41] J. C. Collins and D. E. Soper, Nucl. Phys. B **193**, 381 (1981) [Erratum-ibid. B **213**, 545 (1983)].

[42] J. C. Collins and D. E. Soper, Nucl. Phys. B **194**, 445 (1982).

[43] J. C. Collins and D. E. Soper, Nucl. Phys. B **197**, 446 (1982).

[44] J. C. Collins, D. E. Soper and G. Sterman, Nucl. Phys. B **250**, 199 (1985).

[45] J. C. Collins and J. W. Qiu, Phys. Rev. D **39**, 1398 (1989).

[46] J. C. Collins, D. E. Soper and G. Sterman, "Factorization Of Hard Processes In QCD," Adv. Ser. Direct. High Energy Phys. **5**, 1 (1988) [arXiv:hep-ph/0409313].

[47] J. C. Collins, Nucl. Phys. B **396**, 161 (1993) [arXiv:hep-ph/9208213].

[48] J. C. Collins, arXiv:hep-ph/9510276.

[49] J. C. Collins, arXiv:hep-ph/9705393.

[50] J. C. Collins, Phys. Lett. B **536**, 43 (2002) [arXiv:hep-ph/0204004].

[51] J. C. Collins, Acta Phys. Polon. B **34**, 3103 (2003) [arXiv:hep-ph/0304122].

[52] H. Contopanagos, E. Laenen and G. Sterman, Nucl. Phys. B **484**, 303 (1997) [arXiv:hep-ph/9604313].

[53] J. S. Conway *et al.*, Phys. Rev. D **39** (1989) 92.

[54] D. de Florian and M. Grazzini, Nucl. Phys. B **616**, 247 (2001) [arXiv:hep-ph/0108273].

[55] E. Di Salvo, AIP Conf. Proc. **747**, 215 (2005) [arXiv:hep-ph/0407208].

[56] DØ Collaborarion
`http://www-d0.fnal.gov/Run2Physics/WWW/results/prelim/EW/E13/E13.pdf`
Accessed, July 26, 2005

[57] Y. L. Dokshitzer, D. Diakonov and S. I. Troian, Phys. Rept. **58**, 269 (1980).

[58] J. T. Donohue and S. A. Gottlieb, Phys. Rev. D **23**, 2577 (1983) [Phys. Rev. D **23**, 2581 (1983)].

[59] S. D. Drell and T. M. Yan, Phys. Rev. Lett. **25**, 316 (1970) [Erratum-ibid. **25**, 902 (1970)].

[60] S. D. Drell and T. M. Yan, Annals Phys. **66**, 578 (1971) [Annals Phys. **281**, 450 (2000)].

[61] S. Eidelman *et al.* [Particle Data Group], Phys. Lett. B **592**, 1 (2004).

[62] R. K. Ellis, H. Georgi, M. Machacek, H. D. Politzer and G. G. Ross, Nucl. Phys. B **152**, 285 (1979).

[63] R. K. Ellis, D. A. Ross and S. Veseli, Nucl. Phys. B **503**, 309 (1997) [arXiv:hep-ph/9704239].

[64] R. K. Ellis and S. Veseli, Nucl. Phys. B **511**, 649 (1998) [arXiv:hep-ph/9706526].

[65] K. J. Eskola, P. Hoyer, M. Vanttinen and R. Vogt, Phys. Lett. B **333**, 526 (1994) [arXiv:hep-ph/9404322].

[66] G. Fai, J-W. Qiu and X-F. Zhang, Phys. Rev. C **71**, 014901 (2005) [arXiv:hep-ph/0409003].

[67] S. Falciano *et al.* [NA10 Collaboration], Z. Phys. C **31**, 513 (1986).

[68] R. P. Feynman, Phys. Rev. Lett. **23**, 1415 (1969).

[69] R. P. Feynman, "Photon-Hadron Interactions,"

[70] L. P. Gamberg and G. R. Goldstein, AIP Conf. Proc. **792**, 941 (2005) [arXiv:hep-ph/0509312].

[71] K. Gottfried and J. D. Jackson, Nuovo Cim. **33**, 309 (1964).

[72] I. S. Gradshteyn and I. M. Ryzhik. A. Jeffrey, Editor "Table of integrals, series and products", Fifth Edition. *New York, USA: Academic Press (1994) 1204p*

[73] W. Greiner and A. Schaefer, "Quantum chromodynamics," *Berlin, Germany: Springer (1994) 414 p*

[74] D. J. Gross and F. Wilczek, Phys. Rev. D **8**, 3633 (1973).

[75] D. J. Gross and F. Wilczek, Phys. Rev. D **9**, 980 (1974).

[76] D. J. Gross, Proc. Nat. Acad. Sci. **102**, 9099 (2005) [Int. J. Mod. Phys. A **20**, 5717 (2005 RMPHA,77,837-849.2005)].

[77] M. Guanziroli *et al.* [NA10 Collaboration], Z. Phys. C **37** (1988) 545.

[78] F. Halzen and A. D. Martin, "Quarks And Leptons: An Introductory Course In Modern Particle Physics" *New York, USA: Wiley (1984) 396p*

[79] S. W. Herb *et al.*, Phys. Rev. Lett. **39**, 252 (1977).

[80] J. L. Hewett, Phys. Rev. Lett. **82**, 4765 (1999) [arXiv:hep-ph/9811356].

[81] X. d. Ji and F. Yuan, Phys. Lett. B **543**, 66 (2002) [arXiv:hep-ph/0206057].

[82] H. F. Jones and J. Wyndham, Nucl. Phys. B **176**, 466 (1980).

[83] A. Kulesza and W. J. Stirling, J. Phys. G **26**, 637 (2000) [arXiv:hep-ph/9912300].

[84] Z. Kunszt and D. E. Soper, Phys. Rev. D **46**, 192 (1992).

[85] E. Laenen, G. Sterman and W. Vogelsang, Phys. Rev. D **63**, 114018 (2001) [arXiv:hep-ph/0010080].

[86] C. S. Lam and W. K. Tung, Phys. Rev. D **18** (1978) 2447.

[87] C. S. Lam and W. K. Tung, Phys. Lett. B **80**, 228 (1979).

[88] C. S. Lam and W. K. Tung, Phys. Rev. D **21** (1980) 2712.

[89] F. Landry, R. Brock, P. M. Nadolsky and C. P. Yuan, Phys. Rev. D **67**, 073016 (2003) [arXiv:hep-ph/0212159].

[90] E. Leader and E. Predazzi, "An Introduction to gauge theories and modern particle physics. Vol. 1: Electroweak interactions, the new particles and the parton model," Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **3**, 1 (1996).

[91] E. Leader and E. Predazzi, “An Introduction to gauge theories and modern particle physics. Vol. 2: CP Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **4**, 1 (1996).

[92] S. B. Libby and G. Sterman, “Jet And Lepton Pair Production In High-Energy Lepton - Hadron And Hadron - Phys. Rev. D **18**, 3252 (1978).

[93] S. B. Libby and G. Sterman, Phys. Rev. D **18**, 4737 (1978).

[94] Z. Lu and B. Q. Ma, Phys. Lett. B **615**, 200 (2005) [arXiv:hep-ph/0504184].

[95] P. L. McGaughey, J. M. Moss and J. C. Peng, Ann. Rev. Nucl. Part. Sci. **49**, 217 (1999) [arXiv:hep-ph/9905409].

[96] E. Mirkes and J. Ohnemus, Phys. Rev. D **51**, 4891 (1995) [arXiv:hep-ph/9412289].

[97] A. H. Mueller, “Cut Vertices And Their Renormalization: A Generalization Of The Wilson Phys. Rev. D **18**, 3705 (1978).

[98] T. Muta, World Sci. Lect. Notes Phys. **57**, 1 (1998).

[99] S. Palestini *et al.*, Phys. Rev. Lett. **55**, 2649 (1985).

[100] J. C. Peng *et al.* [E866/NuSea Collaboration], Phys. Rev. D **58**, 092004 (1998) [arXiv:hep-ph/9804288].

[101] J. C. Peng *et al.*, AIP Conf. Proc. **870**, 681 (2006).

[102] M. E. Peskin and D. V. Schroeder, “An Introduction To Quantum Field Theory,”

[103] H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973).

[104] J. Pumplin, D. R. Stump, J. Huston, H. L. Lai, P. Nadolsky and W. K. Tung, JHEP **0207**, 012 (2002) [arXiv:hep-ph/0201195].

[105] J-W. Qiu
“Predictive power of perturbative QCD”, Light-Front Quantization and Perturbative QCD
`http://www.iitap.iastate.edu/reports/lfw/pdf/ProPart9.pdf`
Accessed, August 14, 2006

[106] J. w. Qiu and X. f. Zhang, Phys. Rev. Lett. **86**, 2724 (2001) [arXiv:hep-ph/0012058].

[107] J. w. Qiu and X. f. Zhang, Phys. Rev. D **63**, 114011 (2001) [arXiv:hep-ph/0012348].

[108] J. w. Qiu, R. Rodriguez and X. f. Zhang, Phys. Lett. B **506**, 254 (2001) [arXiv:hep-ph/0102198].

[109] J. P. Ralston and D. E. Soper, Nucl. Phys. B **152**, 109 (1979).

[110] J. L. Rosner, Phys. Rev. D **54**, 1078 (1996) [arXiv:hep-ph/9512299].

[111] D. E. Soper, Phys. Rev. Lett. **43**, 1847 (1979).

[112] D. E. Soper, Nucl. Phys. Proc. Suppl. **53**, 69 (1997) [arXiv:hep-lat/9609018].

[113] D. E. Soper, arXiv:hep-ph/0011256.

[114] G. Sterman, Nucl. Phys. B **281**, 310 (1987).

[115] G. Sterman, "An Introduction to quantum field theory,"

[116] G. Sterman, arXiv:hep-ph/9606312.

[117] G. Sterman and W. Vogelsang, JHEP **0102**, 016 (2001) [arXiv:hep-ph/0011289].

[118] G. Sterman, arXiv:hep-ph/0412013.

[119] G. Sterman, Acta Phys. Polon. B **36**, 389 (2005) [arXiv:hep-ph/0410014].

[120] G. Sterman, "Resummation W/Z", CTEQ 2006 Summer School.
`http://www.phys.psu.edu/~cteq/schools/summer06/sterman/`
Accessed, March 13, 2007

[121] R. D. Tangerman and P. J. Mulders, Phys. Rev. D **51**, 3357 (1995) [arXiv:hep-ph/9403227].

[122] R. S. Towell [FNAL E906 Collaboration], AIP Conf. Proc. **747**, 211 (2005).

[123] A. Vogt, Phys. Lett. B **497**, 228 (2001) [arXiv:hep-ph/0010146].

[124] F. Wilczek, arXiv:hep-th/9609099.

[125] F. Wilczek, Proc. Nat. Acad. Sci. **102**, 8403 (2005) [Int. J. Mod. Phys. A **20**, 5753 (2005 RMPHA,77,857-870.2005)] [arXiv:hep-ph/0502113].

[126] T. M. Yan, arXiv:hep-ph/9810268.

[127] C-P. Yuan
"Drell-Yan type processes", CTEQ Summer School, June 3-5, 2002
`http://www.phys.psu.edu/~cteq/schools/summer02/yuan/`
Accessed, September 12, 2006

[128] L. Y. Zhu *et al.* [FNAL-E866/NuSea Collaboration], "Measurement of angular distributions of Drell-Yan dimuons in p + d arXiv:hep-ex/0609005.

ACKNOWLEDGEMENTS

I would like to express my gratitude to those who helped me during the research and the writing of this dissertation.

Dr. Jianwei Qiu I thank you for your guidance and support. Your patience was almost unlimited and you hope in me against all hope. You trust me and treat me with confidence and friendship. You taught me to look at *the big picture* and you encourage me when I was about to give up. Education is more than the passing of knowledge. You gave me lessons about teamwork and how to collaborate. They will not be forgotten.

I wish to express my indebtedness to Dr.Eli Rosenberg, chairman of the department, for his support and help during my time at Iowa State.

I would like also to thank Fred Olness in particular and the CTEQ collaboration in general. Dr. Olness provided the loadCTEQ5.m code to simulate the parton distribution functions in proton used in all the graphs of this document. The CTEQ collaboration and its summer schools are an invaluable source of learning and exchange of ideas.